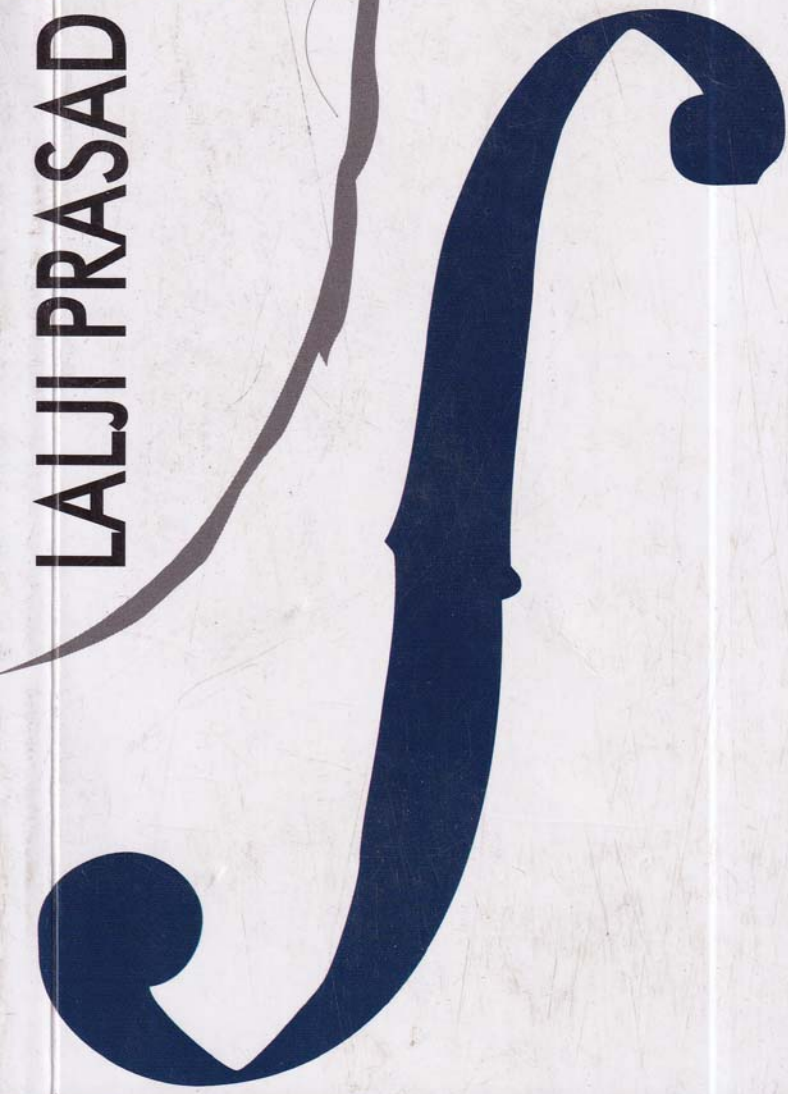
NEW EDITION

Based on the U.G.C. Model Curriculum

In Accordance with +3 Syllabus for Subsidiary + General + Honours

LALJI PRASAD



U.G.C. MODEL CURRICULUM

DIFFERENTIAL CALCULUS

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LIST OF IMPORTANT FORMULAE

I. Important Limits :

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1$, where x is a radian measure

(ii) $\underset{n\to \pm\infty}{\text{Lt}} \left(1+\dfrac{1}{n}\right)^n = e$ or $\underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = e$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x)}{x} = 1$

(iv) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 1}{x} = 1.$

(v) $\underset{x\to 0}{\text{Lt}} \dfrac{a^x - 1}{x} = \log_e a$

(vi) $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x} = n.$

(vii) $\underset{x\to \infty}{\text{Lt}} \dfrac{x^n}{n} = 0.$

(viii) $\underset{x\to \infty}{\text{Lt}} x^n = 0 (-1 < x < 1).$

II. Standard Differentiation :

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

2. $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

3. $\dfrac{d}{dx}(e^n) = e^x$

4. $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

5. $\dfrac{d}{dx}(\sin x) = \cos x$

6. $\dfrac{d}{dx}(\cos x) = -\sin x$

7. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

8. $\dfrac{d}{dx}(\cot x) = -\text{cosec}^2 x$

9. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

10. $\dfrac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

11. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

12. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

13. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

14. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

15. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

16. $\dfrac{d}{dx}(\text{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{x^2-1}}$

17. $\dfrac{d}{dx}(\sin bx) = \cos bx$

18. $\dfrac{d}{d}(\cos bx) = \sin bx$

19. $\frac{d}{dx}(\tan bx) = \sec b^2 x$

20. $\frac{d}{dx}(\cot bx) = -\operatorname{cosec} b^2 x$

21. $\frac{d}{dx}(\sec bx) = \sec bx \tan bx$

22. $\frac{d}{dx}(\operatorname{cosec} bx) = -\operatorname{cosec} bx \cot bx$

23. $\frac{d}{dx}(\sin b^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

24. $\frac{d}{dx}(\operatorname{cosec} b^{-1} x) = -\frac{1}{x\sqrt{x^2+1}}$

III. The n^{th} derivative of some special functions

1. $\frac{d^m}{dx^n}(x^n) = n!$

2. $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}$ (m being a + ve integer $> n$)

3. $\frac{d^n}{dx^n}(e^x) = e^x$

4. $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

5. $\frac{d^n}{dx^n}\frac{1}{x+a} = \frac{(-1)^n n!}{(x+a)^{n+1}}$

6. $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

7. $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

8. $\frac{d^n}{dx^n}\sin ax = a^n \sin\left(\frac{n\pi}{2} + ax\right)$

9. $\frac{d^n}{dx^n}\cos ax = a^n \cos\left(\frac{n\pi}{2} + ax\right)$

10. $\frac{d^n}{dx^n}(e^{ax}\sin bx) = (a^2+b^2)^{\frac{n}{2}}\, e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$

11. $\frac{d_n}{dx^n}(e^{ax}\cos bx) = (a^2+b^2)^{\frac{n}{2}}\, e^{ax}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right)$

12. $\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n\, n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta,$

where $\theta = \tan^{-1}\frac{a}{x}$.

13. $\frac{d^n}{dx^n}(\tan^{-1}x) = (-1)^{n-1}\,(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$

14. **Leibnitz's Theorem**

$$(uv)_n = u_n v + {}^nC_1 u_{n-1}\, v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + v u_n$$

IV. (a) Taylor's series (extended to infinity)

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!}f''(x) + \ldots \text{ to } \infty$$

(b) Machaurin's series (extended to infinity)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots \text{ to } \infty$$

(c) Expansions of some well-known functions in series

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$ for all values of x

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$ for all values of x

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$ for all values of x

(iv) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

(v) $\sin bx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$ for all values of x

(vi) $\cos bx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$ for all values of x

(vii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \text{ for } -1 < x \leq 1$

(viii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \text{ for } -1 < x < 1$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \text{ for } -1 < x < 1$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \text{ for } -1 < x < 1$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \text{ for } -1 < x < 1$

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots \text{ for } -1 < x < 1$

$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ldots \text{for } -1 < x < 1.$

(ix) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots \text{for } -1 \leq x \leq 1.$

(x) $\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \ldots \text{for } -1 \leq x \leq 1.$

V. Tangent and Normal

(i) Equation of the tangent

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad (X - x)f_x + (Y - y)f_y = 0$$

(ii) Equation of the normal

$$Y - y = -\frac{dx}{dy}(X - x) \quad \text{or} \quad \frac{X - x}{f_x} = \frac{Y - y}{f_y}$$

(iii) Cartesian sub-tangent $= \frac{y}{y_1}$; sub-normal $= yy_1$

$$\text{Length of tangent} = \frac{y\sqrt{1 + y_1^2}}{y_1}$$

$$\text{Length of normal} = y\sqrt{1 + y_1^2}$$

(iv) $\tan\psi = \dfrac{dy}{dx}$, $\sin\psi = \dfrac{dy}{ds}$, $\cos\psi = \dfrac{dx}{ds}$

$(dx)^2 = (dx)^2 + (dy)^2$

$\left(\dfrac{ds}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$

(v) $\tan\phi = r\dfrac{d\theta}{dr}$, $\sin\phi = r\dfrac{d\theta}{ds}$, $\cos\phi = \dfrac{dr}{ds}$

$\psi = \theta + \phi$, $(ds)^2 = (dr)^2 + (rd\theta)^2$

$\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2$

(vi) Polar sub-tangent $= r^2\dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$, when $r = \dfrac{1}{u}$

Polar sub normal $= \dfrac{dr}{d\theta}$

(vii) Perpendicular from pole on the tangent (p)

$p = r\sin\phi$

$\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2 = u^2 + \left(\dfrac{du}{d\theta}\right)^2$

(viii) Pedal equation of some well-known curves

(a) Circle $x^2 + y^2 = a^2$ (centre) ... $r = p$.

(b) Parabola $y^2 = 4ax$ (focus) ... $p^2 = ar$

(c) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (focus) ... $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$

(d) Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (focus) ... $\dfrac{b^2}{p^2} = \dfrac{2a}{r} + 1$

(e) Rect. Hyperbola $x^2 - y^2 = a^2$ (centre) ... $pr = a^2$

(f) Cardiode $r = a(1 \pm \cos\theta)$ (pole) ... $r^2 = 2ap^2$

(g) Lemniscate $\left.\begin{matrix} r^2 = a^2\cos 2\theta \\ r^2 = a^2\sin 2\theta \end{matrix}\right\}$ ... $r^3 = a^2 p$

(h) $\left.\begin{matrix} r^n = a^n\cos n\theta \\ r^n = a^n\sin n\theta \end{matrix}\right\}$ (pole) ... $r^{n+1} = a^n p$.

VI. Curvature

1. $\rho = \dfrac{ds}{d\psi}\,[s + f(\psi)]$

2. $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y^2}\,[y = f(x)],\ y_2 \neq 0$

3. $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}\,[x = \phi(t),\ y = \psi(t)]$

4. $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}\,[r = f(\theta)]$

5. $\rho = r\dfrac{dr}{dp}\,[p = f(r)]$

6. $\rho = p + \dfrac{d^2p}{d\psi^2}\,[p = f(\psi)]$

7. $\rho = \text{Lt}\dfrac{x^2}{2y}$ [at the origin, x-axis ($y = 0$) being tangent]

8. $\rho = \text{Lt}\dfrac{y^2}{2x}$ [at the origin, y-axis ($x = 0$) being tangent]

9. Chord of curvature through the pole $= 2\rho \sin\phi$

10. Chord of curvature parallel to x-axis $= 2\rho \sin\psi$

11. Chord of curvature parallel to y-axis $= 2\rho \cos\psi$

12. Centre of curvature $\bar{x} = x - \dfrac{y_1(1 + y_1^2)}{y_2},\ \bar{y} = y + \dfrac{1 + y_1^2}{y_2}\,[y = f(x)]$.

VII. Asymptotes

Let $f(x, y) = 0$ be the equation of a curve.

Method : Let $x^n \phi_n\left(\dfrac{y}{x}\right)$ denote the homogeneous function of degree n in x and y of the curve $f(x, y) = 0$.

Let $y = mx + c$ be the equation of an asymptote of the curve.

Then the values of m and c are obtained from the following equations

$$\left.\begin{aligned} &\phi_n(m) = 0 \\ &c\phi'_n(m) + \phi_{n-1}(m) = 0 \end{aligned}\right\}$$

These equations are obtained by putting $x = 1$ and $y = m$ in the first two highest homogenous functions.

If c is of the form $\frac{0}{0}$, then the values of c are obtained by the equation

$$\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0.$$

VIII. Concavity, Convexity, points of inflexion.

The function $y = f(x)$ is concave upwards (i.e. convex downwards) or concave downwards at the point $x = c$ according as $f'(c) > 0$ or $f''(c) < 0$.

The curve has inflexion at $x = c$ if $f''(c) = 0$ and $f'''(c) \neq 0$.

IX. singular points

To find the multiple points we must have such values of x and y which simultaneously satisfy the three equations

$$f(x, y) = 0,\ f_x(x, y) = 0,\ f_y(x, y) = 0.$$

The point (x, y) will be a double point and will be a node, cusp or conjugate according as

$$(f_{xy})^2 - f_{xx}f_{yy} > 0; = 0 \text{ or } < 0.$$

X. Envelopes

Let $f(x, y, \alpha)$ be a family of curves, α being the parameter.

Eliminate α between $f(x, y, \alpha) = 0$ and $\frac{\delta}{\delta\alpha} f(x, y, \alpha) = 0$.

The α-eliminant is the required envelope.

XI. Maxima and Minima

The necessary and sufficient conditions (in the case of a function of one variable) are

1. For max. or min. $\frac{dy}{dx} = 0$ (necessary)

2. For (i) max. $\frac{d^2y}{dx^2} = -\text{ve}$

   (ii) min. $\frac{d^2y}{dx^2} = +\text{ve}$ (sufficient)

   In the case of a function of two variables i.e., $u = f(x, y)$,

   we, first find out the value of x and y by solving $\frac{\delta u}{\delta x} = 0, \frac{\delta u}{\delta y} = 0.$

   Let $x = a$ and $y = b$ be one such values.

   Then we calculate $f = \frac{\delta^2 u}{\delta x^2}, S = \frac{\delta^2 u}{\delta x \delta y}, t = \frac{\delta^2 u}{\delta y^2}.$

1. If $rt - s^2 > 0$ and

   If $r < 0$, then $f(x, y)$ will be a maximum at (a, b),

   If $r > 0$, then $f(x, y)$ will be a minimum at (a, b)

2. If $rt - s^2 < 0$, then $f(a, b)$ is not an extreme point.

3. If $rt - s^2 = 0$, the case is doubtful.

□

Limits and Continuity

1.1 Introduction

In the present chapter we shall introduce the important concept of limit and continuity of a function whose domain is an interval and whose range is a subset of R.

At the outset, therefore we shall define limit and then proceed to define continuity of a function and establish some important theorems concerning them. But before doing so, we would like to define and enumerate the basic properties of a bounded function.

1.2 Bounded Function

Let $f(x)$ be a function defined over an interval $[a, b]$.

The set of all functional values of $f(x)$ defined over $[a, b]$ will form a set, say S. If the set S is bounded, then we say that $f(x)$ is bounded.

In fact, the entire idea of bounds of $f(x)$ reduces to the discussion of the bounds of the set S which we have done earlier. We know that a set S is said to be bounded above if there exists a number K such that $x < K$ for all $x \in S$.

In a similar manner we can say that a function $f(x)$ is bounded if there exists a positive number K such that $|f(x)| < K$, for every point $x \in [a, b]$ and K is called an upper bound.

If a function $f(x)$ defined over $[a, b]$ is bounded above, then there exists a number M which is the least of all upper bounds. This number M is called the *least upper bound* or simply l.u.b. of $f(x)$ in $[a, b]$.

The l.u.b. M has the following two properties ;

(i) $f(x) \leq M$ for all $x \in [a, b]$;

(ii) corresponding to a given positive number ε, however small, there exists at least one point $\alpha \in [a, b]$ such that

$$M < f(\alpha) + \varepsilon.$$

These two properties of l.u.b. correspond exactly to the two properties of l.u.b. in the case of a set S which we have studied earlier.

Similarly if a function $f(x)$ defined in $[a, b]$ is bounded below, then there exists a number m which is the *greatest lower bound* of $f(x)$ in $[a, b]$.

The g.l.b. has the following properties :

(i) $f(x) \geq m$ for all $x \in [a, b]$;

(ii) corresponding to a given positive number ε, however small, there exists at least one point $\beta \in [a, b]$ such that

$$M > f(\beta) - \varepsilon.$$

Note The number $M - m$ is called the oscillation of $f(x)$ in $[a, b]$.

1.3 $\varepsilon - \delta$: Definition of Limit

The number A is said to be the limit $f(x)$ at $x = a$ if for any arbitrary chosen positive number ε, however small, there exists a corresponding number $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for all values of x for which $0 < |x - a| < \delta$.

1.4 Meaning of $|x - a| < \delta$

We know $|x - a|$ means the absolute value of $x - a$ without regard to sign.

i.e. $|x - a| = x - a$ if $x > a$

$= a - x$ if $x < a$

Thus $|x - a| < \delta \Rightarrow x - a < \delta$ if $x > a$

$\Rightarrow x < a + \delta$

and $|x - a| < \delta \Rightarrow a - x < \delta$ if $x < a$

$\Rightarrow a - \delta < x.$

Hence $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$

$\Rightarrow]a - \delta, a + \delta[.$

Also $0 < |x - a| \Rightarrow x \neq a.$

Thus $0 < |x - a| < \delta \Rightarrow$ deleted nbd. $]a - \delta, a + \delta[.$

1.5 Right hand and Left hand limits

If x approaches a from the right, that is, from larger values of x than a, the limit of $f(x)$ as defined above is called the right hand limit

of $f(x)$ and is written as $\underset{x \to a+0}{\text{Lt}} f(x)$ or $f(a+0)$.

The working rule for finding the right hand limit is the following :

We put $a+h$ for x in $f(x)$ and make $h \to 0$.

In short, we have $f(a+0) = \underset{h \to 0}{\text{Lt}} f(a+h)$.

Similarly if $x \to a$ from the left, that is, from smaller values of x than a, the limit of $f(x)$ is called the left hand limit and is written as

$$\underset{x \to a-0}{\text{Lt}} f(x) \text{ or } f(a-0).$$

In this case we have $f(a-0) = \underset{h \to 0}{\text{Lt}} f(a-h)$.

If both right hand and left hand limits of $f(x)$ as $x \to a$ exist and are equal in value, then we say that the limit of $f(x)$ as $x \to a$ exists and their common value is the limit of the function.

If both of these limits exist but are not equal in value, then we say that the limit of $f(x)$ as $x \to a$ does not exist.

Also, if either or both of these limits do not exist, the limit of $f(x)$ as $x \to a$ does not exist.

Note From the definition of the limit of a function we can deduce the following important criterion for the existence of the limit of a function.

Let $f(x) \to A$ as $x \to a$.

Then from the definition, given $\varepsilon > 0$, however small, we can choose a positive number δ such that

$$|f(x) - A| < \frac{\varepsilon}{2} \text{ whenever } 0 < |x-a| < \delta \qquad ...(1)$$

Now since (1) holds for every value of x in the range $0 < |x-a| < \delta$ we can choose two values of x, say x_1 and x_2 for which (1) holds. Then we have

$$|f(x_1) - A| < \frac{\varepsilon}{2} \text{ and } |f(x_2) - A| < \frac{\varepsilon}{2} \qquad ...(2)$$

whenever $0 < |x_1 - a| < \delta$ and $0 < |x_2 - a| < \delta$.

Now $|f(x_2) - f(x_1)| = |\{f(x_2) - A\} + \{A - f(x_1)\}|$

$$\leq |f(x_2) - A| + |A - f(x_1)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence if $f(x)$ converges to a finite limit A, as $x \to a$, then corresponding to a given $\varepsilon > 0$, however small, there exists a positive number δ such that $|f(x_2) - f(x_1)| < \varepsilon$ whenever for any pair x_1, x_2 of values of x, the inequality $0 < |x - a| < \delta$ holds.

1.6 Algebra of Limits

Theorem 1. The limit of a sum of two functions is equal to the sum of the limits.

Proof : Let $\underset{x \to a}{\text{Lt}} f(x) = A$ and $\underset{x \to a}{\text{Lt}} \phi(x) = B$.

Then we have to prove that

$$\underset{x \to a}{\text{Lt}} \{f(x) + \phi(x)\} = \underset{x \to a}{\text{Lt}} f(x) + \underset{x \to a}{\text{Lt}} \phi(x).$$

Since $\underset{x \to a}{\text{Lt}} f(x) = A$, therefore by definition

$$|f(x) - A| < \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_1 \quad \ldots(1)$$

Similarly, $|\phi(x) - B| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$ $\ldots(2)$

Let $\delta = \min(\delta_1, \delta_2)$.

Then from (1), we get $0 < |x - a| < \delta \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$ $\ldots(3)$

and from (2), $0 < |x - a| < \delta \Rightarrow |\phi(x) - B| < \frac{\varepsilon}{2}$ $\ldots(4)$

Now $|f(x) + \phi(x) - A - B| = |(f(x) - A) + (\phi(x) - B)|$

$$< |f(x) - A| + |\phi(x) - B|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < |x - a| < \delta$.

Hence $\underset{x \to a}{\text{Lt}} \{f(x) + \phi(x)\} = A + B = \underset{x \to a}{\text{Lt}} f(x) + \underset{x \to a}{\text{Lt}} \phi(x)$.

Note (i) This theorem can evidently be extended to any finite number of functions.

(ii) In the similar way, we can prove that

$$\underset{x \to a}{\text{Lt}} \{f(x) - \phi(x)\} = \underset{x \to a}{\text{Lt}} f(x) - \underset{x \to a}{\text{Lt}} \phi(x).$$

Theorem 2. The limit of a product of two functions is equal to the product of the limits.

Here we want to prove that

$$\lim_{x \to a} \{f(x) \cdot \phi(x)\} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \phi(x).$$

Proof : Let $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$.

Now $|f(x)\,\phi(x) - AB|$

$$= |f(x)\,\phi(x) - A\phi(x) + A\phi(x) - AB|$$

$$\leq |f(x)\,\phi(x) - A\phi(x)| + |A\phi(x) - AB|$$

$$= |\phi(x)| \cdot |f(x) - A| + |A| \cdot |\phi(x) - B|.$$

It is given that $\lim_{x \to a} \phi(x) = B$, therefore $\phi(x)$ is bounded in the neighbourhood of $x = a$.

Hence $|\phi(x)| < M$ for all values of x such that $0 < |x - a| < \delta'$.

Then $|f(x)\,\phi(x) - AB| < M\,|f(x) - A| + |A|\,|\phi(x) - B|$.

Since $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$, therefore corresponding to any $\varepsilon > 0$, we can choose a positive number $\delta < \delta'$ such that

$$|f(x) - A| < \frac{\varepsilon}{2M} \text{ and } |\phi(x) - B| < \frac{\varepsilon}{2|A|}$$

whenever $0 < |x - a| < \delta$.

Thus $|f(x)\,\phi(x) - AB| < M\frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2|A|} = \varepsilon$

whenever $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} f(x)\,\phi(x) = AB = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \phi(x)$.

The above theorem can be extended to any finite number of functions.

Theorem 3. The limit of a quotient of two functions is equal to the quotient of the limits, provided the limit of the denominator $\neq 0$.

Proof : Let $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$.

Now $\left|\frac{f(x)}{\phi(x)} - \frac{A}{B}\right| = \left|\left\{\frac{f(x)}{\phi(x)} - \frac{f(x)}{B}\right\} + \left\{\frac{f(x)}{B} - \frac{A}{B}\right\}\right|$

$$= \left|f(x)\left\{\frac{1}{\phi(x)} - \frac{1}{B}\right\} + \frac{1}{B}\{f(x) - A\}\right|$$

$$= \left| \frac{f(x)}{B\phi(x)} \{B - \phi(x)\} + \frac{1}{B} \{f(x) - A\} \right|$$

$$\leq \frac{|f(x)|}{|B|\,|\phi(x)|} |B - \phi(x)| + \frac{1}{|B|} |f(x) - A| \quad \ldots(1)$$

By hypothesis, $\underset{x \to a}{\text{Lt}} f(x) = A$ and $\underset{x \to a}{\text{Lt}} \phi(x) = B$.

That is, the limits of $f(x)$ and $\phi(x)$ exist when $x \to a$, hence these functions are bounded in the neighbourhood of $x = a$.

Let M be the upper bound of $f(x)$ and m, the lower bound of $\phi(x)$ so that $|f(x)| < M$ and $|\phi(x)| > m$.

We may then write (1) as follows :

$$\left| \frac{f(x)}{\phi(x)} - \frac{A}{B} \right| \leq \frac{M}{m|B|} |B - \phi(x)| + \frac{1}{|B|} |f(x) - A| \quad \ldots(2)$$

Since $\underset{x \to a}{\text{Lt}} f(x) = A$ and $\underset{x \to a}{\text{Lt}} \phi(x) = B$, therefore corresponding to $\varepsilon > 0$, we can find positive numbers δ_1, δ_2 such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < |B| \cdot \frac{\varepsilon}{2}$$

and $$0 < |x - a| < \delta_2 \Rightarrow |\phi(x) - B| < \frac{m|B|}{M} \cdot \frac{\varepsilon}{2}.$$

Let $\delta = \min. (\delta_1, \delta_2)$.

Then $$0 < |x - a| < \delta \Rightarrow |f(x) - A| < |B| \cdot \frac{\varepsilon}{2}$$

and $$0 < |x - a| < \delta \Rightarrow |\phi(x) - B| < \frac{m|B|}{M} \cdot \frac{\varepsilon}{2}.$$

Hence from (2), we find that

$$\left| \frac{f(x)}{\phi(x)} - \frac{A}{B} \right| < \frac{M}{m|B|} \cdot \frac{m|B|}{M} \cdot \frac{\varepsilon}{2} + \frac{1}{|B|} \cdot |B| \cdot \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Hence $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \frac{A}{B} = \frac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} \phi(x)}$ provided $B = \underset{x \to a}{\text{Lt}} \phi(x) \neq 0$.

1.7 Theorem

Suppose that there is a $\delta > 0$ such that $f(x) = 0$ whenever $0 < |x - a| < \delta$. Then $\lim_{x \to a} f(x) = 0$.

Proof : According to the statement of the theorem, given $\varepsilon > 0$ we can choose a positive number δ such that

$$0 < |x - a| < \delta \Rightarrow f(x) = 0 \Rightarrow |f(x) - 0| < \varepsilon.$$

Hence $\lim_{x \to a} f(x) = 0$.

Cor. : *Suppose that there is a $\delta > 0$ such that $f(x) = g(x)$ whenever*

$$0 < |x - a| < \delta.$$

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Proof : Let $h(x) = f(x) - g(x)$.

Then $h(x) = 0$ whenever $0 < |x - a| < \delta$.

Hence from the above theorem, $\lim_{x \to a} h(x) = 0$.

But $\lim_{x \to a} h(x) = \lim_{x \to a} \{f(x) - g(x)\}$

$$= \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Therefore $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Note This theorem emphasises the fact that if two functions agree on some nbd. of a point a, then they cannot approach different limits as $x \to a$.

1.8 Theorem

Let f be defined on a domain D and let $f(x) \geq 0$ for all $x \in D$.

If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$.

Proof : Let $\lim_{x \to a} f(x) = A$.

Suppose that A is negative. (so that $-A$ is positive).

Now from the definition of limit,

$|f(x) - A| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Choosing $\varepsilon = -\frac{1}{2}A$, we find that $|f(x) - A| < -\frac{1}{2}A$.

This $\Rightarrow \frac{3A}{2} < f(x) < \frac{A}{2} < 0$ whenever $0 < |x-a| < \delta$.

But it is given that $f(x) \geq 0$ for all $x \in D$.

Hence we get a contradiction.

Consequently A cannot be negative.

Hence $\underset{x \to a}{\text{Lt}} f(x) \geq 0$.

Cor. 1. *Let f be defined on D and let $f(x) > 0$ for all $x \in D$.*

If $\underset{x \to a}{\text{Lt}} f(x)$ exists, then $\underset{x \to a}{\text{Lt}} f(x) \geq 0$.

The proof follows from the fact that $f(x) > 0 \Rightarrow f(x) \geq 0$ and then we apply the above theorem.

Cor. 2. *Let f and g be defined on D and let $f(x) \geq g(x)$ for all $x \in D$.*

Then $\underset{x \to a}{\text{Lt}} f(x) \geq \underset{x \to a}{\text{Lt}} g(x)$ provided these limits exist.

Proof : Let $h(x) = f(x) - g(x)$ so that $h(x) \geq 0$ for all $x \in D$.

Then $\underset{x \to a}{\text{Lt}} h(x) \geq 0$ i.e. $\underset{x \to a}{\text{Lt}} \{f(x) - g(x)\} \geq 0$

This $\Rightarrow \underset{x \to a}{\text{Lt}} f(x) - \underset{x \to a}{\text{Lt}} g(x) \geq 0$ i.e. $\underset{x \to a}{\text{Lt}} f(x) \geq \underset{x \to a}{\text{Lt}} g(x)$.

1.9 Limit as $x \to \infty$ (or $-\infty$).

Definition : A function f is said to approach A as $x \to +\infty$ if corresponding to an arbitrary small positive number ε, there exists a number K such that $|f(x) - A| < \varepsilon$ whenever $x > K$.

Ex.1. Let $f(x) = \frac{1}{x}$, $x \neq 0$.

Here $\underset{x \to \infty}{\text{Lt}} f(x) = 0$.

Ex.2. Let $f(x) = \frac{1}{x^2+1}$.

Here $\underset{x \to \infty}{\text{Lt}} f(x) = 0$.

Ex.3. Let $f(x) = \frac{x^3-1}{x^3+1}$; $x \neq -1$.

Here $\underset{x \to \infty}{\text{Lt}} f(x) = 1$.

1.10 Infinite Limits

Definition : A function f is said to approach $+\infty$ as $x \to a$; in symbols $\lim\limits_{x \to a} f(x) = +\infty$ if given any arbitrary number K, we can choose a +ve number δ such that $f(x) > K$ whenever $0 < |x - a| < \delta$.

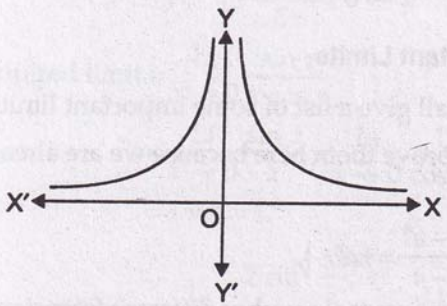
Similarly a function f is said to approach $-\infty$ as $x \to a$; in symbols $\lim\limits_{x \to a} f(x) = -\infty$ if given any arbitrary number K, we can choose a number $\delta > 0$ such that $f(x) < K$ whenever $0 < |x - a| < \delta$.

Ex.1. Let $f(x) = \frac{1}{x^2}; x \neq 0$.

Then $\lim\limits_{x \to 0} f(x) = +\infty$.

That is, $\lim\limits_{x \to 0+0} f(x) = +\infty$ and also $\lim\limits_{x \to 0-0} f(x) = +\infty$.

The graph of this function is shown below :



Ex.2. Let $f(x) = -\frac{1}{x^2}; x \neq 0$.

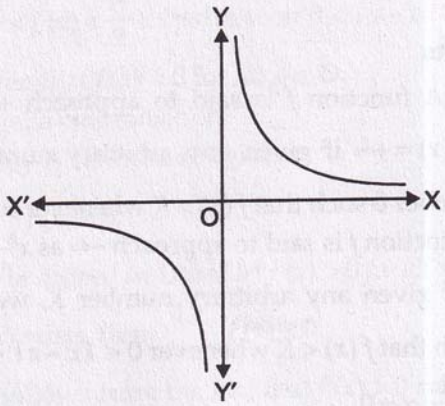
Then $\lim\limits_{x \to 0} f(x) = -\infty$.

The graph of this function is the inverted graph of Ex.1 below x-axis.

Ex.3. Let $f(x) = \frac{1}{x}; x \neq 0$.

Then $\lim\limits_{x \to 0+0} f(x) = +\infty$ and $\lim\limits_{x \to 0-0} f(x) = -\infty$.

The graph of this function is shown below :



Ex.4. $\lim_{x \to 0} \log |x| = -\infty$.

Given $\lambda > 0$, choose $\delta = e^{-\lambda}$.

Now, if $0 < |x - 0| < \delta$, we have $|x| < e^{-\lambda} \Rightarrow \log |x| < -\lambda$ and consequently $\lim_{x \to 0} \log |x| = -\infty$.

1.11 Some Important Limits

Herein we shall give a list of some important limits.

We shall not prove them here because we are already acquainted with them.

(i) $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$,
where n is any real number different from zero.

(ii) $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

(iii) $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

(iv) $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$.

Ex.1. Evaluate : $\lim_{x \to 2} \dfrac{x^5 - 32}{x^3 - 8}$

Soln. The required limit $= \lim_{x \to 2} \dfrac{x^5 - (2)^5}{x^3 - (2)^3} = \lim_{x \to 2} \dfrac{\dfrac{x^5 - (2)^5}{x - 2}}{\dfrac{x^3 - (2)^3}{x - 2}}$

$$= \frac{\underset{x \to 2}{\text{Lt}} \dfrac{x^5 - (2)^5}{x-2}}{\underset{x \to 2}{\text{Lt}} \dfrac{x^3 - (2)^3}{x-2}} = \frac{5 \cdot (2)^{5-1}}{3 \cdot (2)^{3-1}}$$

$$= \frac{5 \times 2^4}{3 \times 2^2} = \frac{5 \times 16}{3 \times 4} = \frac{20}{3}.$$

Ex.2. Evaluate the following :

(i) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 3x}{\sin 5x}$ (ii) $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x}{x}$ (iii) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos 5x}{x^2}$.

Soln. (i) The required limit $= \underset{x \to 0}{\text{Lt}} \dfrac{\dfrac{\sin 3x}{3x} \times 3x}{\dfrac{\sin 5x}{5x} \times 5x}$

$$= \frac{3}{5} \times \frac{\underset{}{\text{Lt}} \dfrac{\sin 3x}{3x}}{\underset{}{\text{Lt}} \dfrac{\sin 5x}{5x}} = \frac{3}{5} \times \frac{1}{1} = \frac{3}{5}.$$

(ii) The required limit $= \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x \cos x}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} \times \underset{x \to 0}{\text{Lt}} \frac{1}{\cos x}$$

$$= 1 \times 1 = 1.$$

(iii) The required limit $= \underset{x \to 0}{\text{Lt}} \dfrac{2 \sin^2 \dfrac{5x}{2}}{x^2} = 2 \underset{x \to 0}{\text{Lt}} \dfrac{\left(\sin \dfrac{5x}{2}\right)^2}{\left(\dfrac{5x}{2}\right)^2 \times \dfrac{4}{25}}$

$$= 2 \times \frac{25}{4} \times \underset{x \to 0}{\text{Lt}} \left(\frac{\sin \dfrac{5x}{2}}{\dfrac{5x}{2}} \right)^2$$

$$= \frac{25}{2} \times 1^2 = 12\frac{1}{2}.$$

Worked Out Examples

Ex.1. Find the limit of the following if it exists : $\underset{x \to 0}{\text{Lt}} \dfrac{1}{1 + e^{1/x}}$.

Soln. Let $f(x) = \dfrac{1}{1+e^{1/x}}$.

Then $f(0+0) = \underset{h \to 0}{\text{Lt}} \dfrac{1}{1+e^{1/(0+h)}} = \underset{h \to 0}{\text{Lt}} \dfrac{1}{1+e^{1/h}}$

$$= \frac{1}{1+e^{\infty}} = \frac{1}{1+\infty} = 0;$$

and $f(0-0) = \underset{h \to 0}{\text{Lt}} \dfrac{1}{1+e^{1/(0-h)}}$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1.$$

Since $f(0+0) \neq f(0-0)$, therefore the limit does not exist.

Ex.2. Evaluate : $\underset{x \to 0}{\text{Lt}} \dfrac{e^{1/x}}{e^{1/x}+1}$.

Soln. R.H.L $= \underset{h \to 0}{\text{Lt}} f(0+h) = \underset{h \to 0}{\text{Lt}} f(h); (h > 0)$

$$= \underset{h \to 0}{\text{Lt}} \frac{e^{1/h}}{e^{1/h}+1} = \underset{h \to 0}{\text{Lt}} \frac{1}{1+e^{-1/h}}.$$

When $h \to 0, e^{-1/h} \to e^{-\infty} = 0$.

$\therefore$ R.H.L. $= \dfrac{1}{1+0} = 1.$

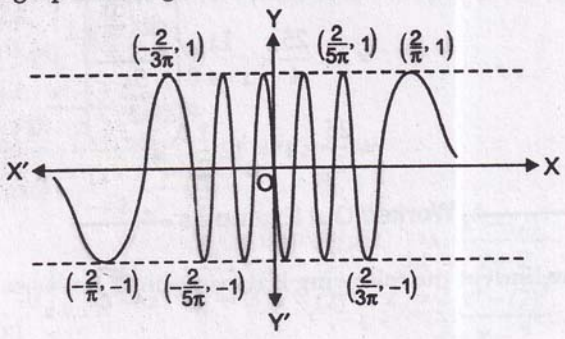
L.H.L $= \underset{h \to 0}{\text{Lt}} f(0-h) = \underset{h \to 0}{\text{Lt}} f(-h) = \underset{h \to 0}{\text{Lt}} \dfrac{e^{-1/h}}{e^{-1/h}+1} = \dfrac{0}{1} = 0.$

Thus R.H.L. $\neq$ L.H.L.

Hence the limit of the function does not exit.

Ex.3. Evaluate : $\underset{x \to 0}{\text{Lt}} \sin \dfrac{1}{x}$.

Soln. The graph of the given function, as we know, is as follows :



We see that as $x \to 0$, the value of $\sin \frac{1}{x}$ oscillates between +1 and −1, passing through zero and intermediate values an infinite number of times.

Evidently there is no definite number to which $\sin \frac{1}{x}$ tends as $x \to 0$ from the right. So the limit on the right does not exist. Similarly the limit on the left does not exist.

Hence the limit of $\sin \frac{1}{x}$ as $x \to 0$ does not exist.

Analytical proof : We know that if the limit exists, then corresponding to any $\varepsilon > 0$, there exists a positive number δ such that $|f(x_2) - f(x_1)| < \delta$ for any two values x_1, x_2 of x in the range $0 < |x - a| < \delta$.

Let us choose $x_2 = \frac{2}{(4n+1)\pi}$ and $x_1 = \frac{2}{(4n-1)\pi}$ where n is a sufficiently large number.

Obviously $x_2 < x_1$ so that we can take $x_1 \leq \delta$.

Also, it is possible (by the Archimedean property of real numbers) to choose n such that

$$(4n-1)\frac{\pi}{2} > \frac{1}{\delta} \text{ i.e. } \frac{2}{(4n-1)\pi} (= x_1) < \delta.$$

Thus we have got a pair of values x_1 and x_2 of x in the range

$$0 < |x| \leq \delta.$$

But for such values of x_2 and x_1 we notice that

$$|f(x_2) - f(x_1)| = \left| \sin\left(\frac{4n+1}{2}\right)\pi - \sin\frac{(4n-1)\pi}{2} \right|$$

$$= \left| \sin\left(2n\pi + \frac{\pi}{2}\right) - \sin\left(2n\pi - \frac{\pi}{2}\right) \right|$$

$$= 2.$$

Since ε can be chosen as small as we please, $|f(x_2) - f(x_1)|$ cannot be made less than ε.

Hence $\underset{x \to 0}{\text{Lt}} \sin \frac{1}{x}$ does not exist.

Similarly by choosing $x_1 = \frac{1}{(2n+1)\pi}$ and $x_2 = \frac{1}{2n\pi}$, we can show

that (i) $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

(ii) $\lim_{x \to 0} \sin \frac{\pi}{x}$ and $\lim_{x \to 0} \cos \frac{\pi}{x}$ do not exist.

Ex.4. Evaluate : $\lim_{x \to 0} x \sin \frac{1}{x}$. **[M.U. 1981]**

Soln. Let $f(x) = x \sin \frac{1}{x}$.

Then $f(0+0) = \lim_{h \to 0} (0+h) \sin \frac{1}{0+h}$

$= \lim_{h \to 0} h \sin \frac{1}{h}$ ...(1)

But $\sin \frac{1}{h}$ is bounded lying between -1 and $+1$.

Hence $f(0+0) = \lim_{h \to 0} h \times$ a finite quantity

$= 0 \times$ a finite quantity

$= 0$.

Similarly, $f(0-0) = \lim_{h \to 0} (0-h) \sin \frac{1}{0-h}$

$= \lim_{h \to 0} h \sin \frac{1}{h} = 0$ as before.

Hence $f(0+0) = f(0-0)$ and therefore the limit of the function exists and $= 0$.

Ex.5. Find $\lim_{x \to 0} \{e^x \text{ sgn } (x + [x])\}$ **where sgn ≡ signum function and** $[x]$ **≡ greatest integer** $\leq x$.

Soln. We have, $\text{sgn } (x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0. \end{cases}$

Also, $[x] = 0$ if $0 \leq x < 1$

$= -1$ if $-1 \leq x < 0$

Now, R.H.L. $= \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} e^h \text{ sgn } (h + [h])$

$\lim_{\to 0} e^h \text{ sgn } (h+0)$; since $h > 0$

$$= \underset{h \to 0}{\text{Lt}}\, e^{h} \operatorname{sgn} h = \underset{h \to 0}{\text{Lt}}\, e^{h}(1) = e^{0} = 1.$$

$$\text{L.H.L.} = \underset{h \to 0}{\text{Lt}}\, f(0-h) = \underset{h \to 0}{\text{Lt}}\, f(-h)$$

$$= \underset{h \to 0}{\text{Lt}}\, e^{-h} \operatorname{sgn}(-h + [-h]) = \underset{h \to 0}{\text{Lt}}\, e^{-h} \operatorname{sgn}(-h-1)$$

$$= \underset{h \to 0}{\text{Lt}}\, e^{-h}(-1);\ \because -(h-1) < 0$$

$$= -1.$$

Since R.H.L. $\neq$ L.H.S., the limit does not exist.

CONTINUITY

1.12 Introduction

In this chapter we shall dwell upon a very important concept of mathematics and that is the continuity of a function. The intuitive idea of continuity is derived from geometrical considerations. If the graph of the function $y = f(x)$ is an unbroken curve we say that the function is continuous. This requires that we should be able to draw the graph of the curve without lifting the pencil from the graph paper. But as we know that it is not possible to draw the graph of every function and so we are in need of a definition for the continuity of a function whether it is amenable to graph or not.

We now give the arithmetical definition of continuity as given by Cauchy which is usually called the $\varepsilon - \delta$ definition of a continuous function.

1.13 Cauchy's definition of continuity

A real-valued function f defined on an open interval I is said to be continuous at $a \in I$ if for any arbitrarily chosen positive number ε, however small, we can find a corresponding number δ such that

$$|f(x) - f(a)| < \varepsilon \qquad \ldots(1)$$

for all values of x for which $|x - a| < \delta$.

In other words, f is continuous at a if given $\varepsilon > 0$, we can find a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

This means that the function f will be continuous at $x = a$ if the difference between $f(a)$ and the value of $f(x)$ at any point in the interval $]a - \delta, a + \delta[$ can be made less than a preassigned positive number ε, that is, for all $x \in]a - \delta, a + \delta[$, $f(x)$ is confined to lie

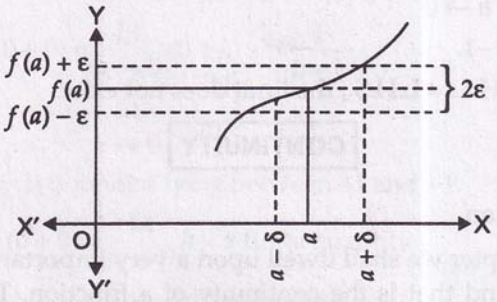
between $f(a) - \varepsilon$ and $f(a) + \varepsilon$ i.e. $f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[$.

Accordingly the statement $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$ is equivalent to the statement

$$x \in]a - \delta, a + \delta[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[.$$

It has to be observed that we choose δ after we have chosen ε.

The following geometrical picture will be helpful in visualising the definition of a continuous function at $x = a$.

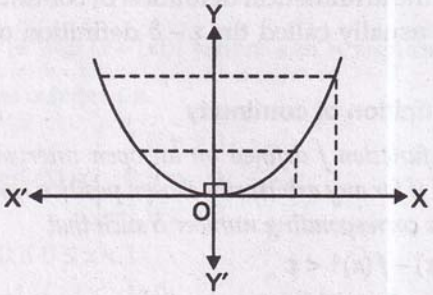


Obs. : We have defined the continuity of a function as follows :

A function f is said to be continuous at a point a if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Now this δ depends in general not only on ε but also on the point a at which the continuity of f is being considered.

For example, we consider the function f defined by $f(x) = x^2$ for all $x \in R$.



We take $\varepsilon = \frac{1}{4}$ and $a = 0$; then $|f(x) - f(a)| < \frac{1}{4}$ is true whenever $|x^2 - 0| < \frac{1}{4}$, i.e. whenever $x < \frac{1}{2}$. Thus we find that in this case $\delta = \frac{1}{2}$ works at $a = 0$ corresponding to $\varepsilon = \frac{1}{4}$.

But it can be easily seen that $\delta = \frac{1}{2}$ does not work at the point $a = 1$ corresponding to $\varepsilon = \frac{1}{4}$.

For if we take $x = 1.4$, then $|x - a| = |1.4 - 1| = 0.4 < \frac{1}{2}$;

but $|f(x) - f(a)| = |x^2 - a^2|$

$= (1.4)^2 - 1^2 = 0.96$ which is $\frac{1}{4}$.

Thus we find that given $\varepsilon > 0$, the same δ does not work for all the points of the domain of the function. In other words, δ depends on the given ε and the particular point chosen. In this sense, we say that δ is a function of ε and x and we may write it as $\delta(\varepsilon, x)$.

Obs. : From the definition of continuity viz.,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

it transpires that the limit of $f(x)$ is $f(a)$ when $x \to a$.

Thus if a function is continuous at $x = a$, it is necessary that

(i) $\underset{x \to a}{\text{Lt}} f(x)$ must exist.

(ii) The function must be defined at $x = a$ and that the aforesaid limit is $f(a)$.

Since the definition of continuity involves the idea of limit, therefore we should have continuity from the left and continuity from the right corresponding to the left hand limit and right hand limit.

1.14 Continuity from the left and continuity from the right

Let f be a function defined on an open interval I and let $a \in I$.

If $\underset{x \to a - 0}{\text{Lt}} f(x)$ exists and is $= f(a)$ we say that the function f is continuous from the left.

Similarly if $\underset{x \to a + 0}{\text{Lt}} f(x)$ exists and is $= f(a)$ we say that the function f is continuous from the right.

But we write $\underset{x \to a - 0}{\text{Lt}} f(x) = f(a - 0)$ and $\underset{x \to a + 0}{\text{Lt}} f(x) = f(a + 0)$.

Hence the function f will be continuous from the left if $f(a - 0) = f(a)$ and the function will be continuous from the right if $f(a + 0) = f(a)$.

The function is said to be continuous at the point $x = a$ if

$$f(a-0) = f(a) = f(a+0).$$

If the function is not continuous at $x = a$, then we say that it is *discontinuous* at that point.

1.15 I. Continuity in an open interval

A function f is said to be continuous in the open interval, say $]a, b[$, if it is continuous at each point $x \in]a, b[$.

II. Continuity in a closed interval

Let f be a function defined on the closed interval $[a, b]$.

We say that f is continuous on the closed interval if

(i) it is continuous on the open interval $]a, b[$;

(ii) it is continuous from the right at a;

(iii) it is continuous from the left at b.

1.16 Working rule to test the continuity

We have shown above that a function is said to be continuous at $a \in I$ if $\lim_{x \to a} f(x)$ exists, is finite and is $= f(a)$.

Thus a function f is continuous at $x = a$ if

$$f(a-0) = f(a) = f(a+0)$$

i.e.

$$\lim_{h \to 0} f(a-h) = f(a) = \lim_{h \to 0} f(a+h).$$

This is a working formula for testing the continuity of a function at a given point.

1.17 Distinction between the value and the limit of a function

At this stage it is worthwhile to throw some light on the distinction between the limit and the value of a function.

By the value of a function $f(x)$ at $x = a$ we mean that value which is obtained by putting $x = a$ in the given function $y = f(x)$. Thus the value of $f(x)$ at $x = a$ is $f(a)$.

In this regard we may have the following different cases :

(i) The value and the limit both exist and are equal.

For example, in the case of the function $f(x) = \cos x$, the limit and the value at any point $x = a$ both exist and are equal to $f(a)$. In fact this function is continuous for all real values of x.

(ii) The value and the limit both exist but are unequal.

For example, $f(x) = \dfrac{\sin x}{x}; x \neq 0$

$f(0) = 0.$

Here the limit of $f(x)$ as $x \to 0$ is 1, whereas the actual value of the function at $x = 0$ is given to be 0. Hence the limit and the value both exist but are unequal. In fact this function is discontinuous at $x = 0$.

(iii) The value exists but the limit does not exist.

For example, consider the function f defined by

$$f(x) = \frac{1}{2} - x, \text{ when } x \leq \frac{1}{2}$$

$$= \frac{3}{2} - x, \text{ when } x > \frac{1}{2}.$$

Here $f\left(\dfrac{1}{2}\right) = 0$ but the limit does not exist since

$$f\left(\frac{1}{2} - 0\right) = 0 \text{ and } f\left(\frac{1}{2} + 0\right) = 1.$$

(iv) The limit exists but the value does not exist.

For example, consider the function $f(x) = \dfrac{x^2 - a^2}{x - a}$.

Here the function is not defined at $x = a$ but the limit exists and is $= 2a$.

(v) The limit and the value both to do not exist.

For example, consider the function $f(x) = \sin \dfrac{1}{x}$.

Here neither the value nor the limit exists at $x = 0$.

1.18 Types of Discontinuities

Let f be a function defined on an interval I. We suppose that f is discontinuous at a point $a \in I$. This discontinuity may be of the following types :

(i) Removable discontinuity : If $\underset{x \to a}{\text{Lt}} f(x)$ exists i.e. if $f(a + 0) = f(a - 0)$ but $\neq f(a)$, then we say that f has removable discontinuity at $x = a$.

(ii) Discontinuity of the first kind : If $f(a + 0)$ and $f(a - 0)$ exist but $f(a + 0) \neq f(a - 0)$, then we say that f has a discontinuity of first kind at $x = a$.

(iii) Discontinuity of the second kind : If neither of the limits $f(a-0)$ and $f(a+0)$ exist, then we say that f has a discontinuity of the second kind at $x=a$.

1.19 Continuity of some elementary functions

Here we shall test the continuity of some elementary functions by taking up the $\varepsilon-\delta$ definition of continuity.

Ex.1. Show that the function $f(x)=x$ is continuous for all real values of x.

Soln. Let $f(x)=x$.

We want to test the continuity of the function at $x=c$.

Now $|f(x)-f(c)| = |x-c|$.

Therefore taking $|x-c|<\delta$, we get $|f(x)-f(c)|<\delta$.

Thus taking $\delta=\varepsilon$ we find that $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$.

Hence $f(x)=x$ is continuous for all $x \in R$.

Ex.2. Show that $\frac{1}{x}$ is continuous for all real values of x except $x=0$.

Soln. Let $f(x)=\frac{1}{x}$ and let $x=c$ be any real number where $c \neq 0$.

$$\text{Now } |f(x)-f(c)| = \left|\frac{1}{x}-\frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \left|\frac{x-c}{cx}\right| \quad \ldots(1)$$

Let $|x-c|<\delta$, so that $c-\delta<x<c+\delta$.

Then from (1), $|f(x)-f(c)| < \dfrac{\delta}{c(c-\delta)}$.

Taking $\varepsilon = \dfrac{\delta}{c(c-\delta)}$ we find that $|f(x)-f(c)|<\varepsilon$ whenever $|x-c|<\delta$.

Hence $f(x)=\frac{1}{x}$ is continuous at $x=c\ (\neq 0)$.

Ex.3. Show that x^m $(m>0)$ is continuous for all $x>0$.

Soln. Here $|f(x)-f(c)| = |x^m-c^m|$

$$= |(c+h)^m-c^m| \text{ where } h \leq \delta.$$

$$\text{Now } (c+h)^m-c^m = c^m\left[\left(1+\frac{h}{c}\right)^m-1\right] \quad \ldots(1)$$

We know that if n be an arbitrary integer $> m$, then

$$1<\left(1+\frac{h}{c}\right)^m<\left(1+\frac{h}{c}\right)^n.$$

Now, when $h \to 0$, $\left(1+\frac{h}{c}\right)^n \to 1$.

Hence $\lim_{h \to 0}\left(1+\frac{h}{c}\right)^m = 1$.

Therefore from (1), $f(c+h)-f(c) \to 0$ as $h \to 0$

which proves that the function $f(x) = x^m$ is continuous for $x > 0$.

Ex.4. Show that sin x is continuous for every real value of x.

Soln. Let $f(x) = \sin x$ and let $x = c$ be any arbitrary point in the interval $0 \le c \le \frac{\pi}{2}$.

Now $|f(x)-f(c)| = |\sin x - \sin c|$

$$= \left|2\cos\frac{x+c}{2}\cdot\sin\frac{x-c}{2}\right| \le 2\left|\cos\frac{x+c}{2}\right|\cdot\left|\sin\frac{x-c}{2}\right| \quad ...(1)$$

But we know that

$$\left|\cos\frac{x+c}{2}\right| \le 1 \text{ and } \left|\sin\frac{x-c}{2}\right| \le \left|\frac{x-c}{2}\right|.$$

Thus from (1), we have $|f(x)-f(c)| < 2\left|\frac{x-c}{2}\right| = |x-c|$.

Hence $|f(x)-f(c)| < \varepsilon$, when $|x-c| < \varepsilon$.

This shows the existence of $\delta = \varepsilon$.

Hence sin x is continuous at $x = c$ when $0 \le c \le \frac{\pi}{2}$.

Ex.5. Show that the function a^x $(a > 0)$ is continuous for all $x \in R$.

Soln. Let $f(x) = a^x$ and let $c \in R$ be an arbitrary point. In order to show that $f(x)$ is continuous at $x = c$, we need to show that

$$|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon.$$

Now $|f(x)-f(c)| = |a^x - a^c|$

$$= |a^{c+h} - a^c|, \text{ where } h \le \delta$$

$$= |a^c(a^h - 1)|. \quad ...(1)$$

We know that if $p > 0$ and n is any positive integer, then

$$(1+p)^n > 1 + np.$$

Hence $p < \frac{(1+p)^n - 1}{n}$.

Assuming that $h \to 0$ and $p = a^h - 1$ in this inequality, we find that

$$a^h - 1 < \frac{a^{nh} - 1}{n} \quad (h > 0, n > 1) \qquad \ldots(2)$$

Let us now choose the number n such that $n \leq \frac{1}{h} < n + 1$.

so that $nh \leq 1$ and $\therefore\ a^{nh} \leq a$ for $a > 2$.

Hence from (2), $a^h - 1 \leq \frac{a-1}{n}$.

Therefore from (1), $|f(x) - f(c)| \leq \left|\frac{a^c(a-1)}{n}\right|$

which can be made as small as we like by taking $n \to \infty$.
(i.e. $h \to 0$).

Hence the function $f(x) = a^x$ is continuous for all $x \in R$.

Note The corresponding inverse function i.e. all logarithmic functions are continuous.

Further Worked Out Examples

Ex.1. Discuss the continuity of the following function at $x = 0$;

$$f(x) = \begin{cases} \frac{\sin x}{x} \text{ when } x \neq 0 \\ 2 \text{ when } x = 0. \end{cases}$$

[P.U. 56H, 62H; Bhag. 90H]

Soln. We know that $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$.

Hence the limit of $f(x)$ as $x \to 0$ is $= 1$.

But it is given that $f(0) = 2$.

Hence the limit of $f(x)$ at $x = 0$ is not the same as the value of $f(x)$ at $x = 0$.

Hence the function is discontinuous at $x = 0$.

Note If we had defined the value of the function, $f(x)$ at $x = 0$ as $f(0) = 1$, then the limit of $f(x)$ as $x \to 0$ becomes $= f(0)$. Hence the function in this case becomes continuous at $x = 0$. Thus if the right hand limit and the left hand limit of the function $f(x)$ at say $x = a$ are equal and this limit $\neq f(a)$, then the discontinuity at $x = a$ may be removed or that the function $f(x)$ may be made continuous at $x = a$ by suitably defining the value of the function at $x = a$.

That is why, this kind of discontinuity in which $f(a + 0) = f(a - 0)$ but $\neq f(a)$ is called a removable discontinuity at $x = a$.]

Ex.2. A function f is defined by the relation

$$f(x)=\frac{\cos ax-\cos bx}{x^2};\ x\neq 0.$$

Define it suitably at $x=0$ so that the function becomes continuous there. [P.U. 64.H]

Soln. We have, $f(x)=\dfrac{\cos ax-\cos bx}{x^2}=\dfrac{2\sin\frac{a+b}{2}x\cdot\sin\frac{b-a}{2}x}{x^2}$

$$=2\cdot\frac{\sin\frac{a+b}{2}x}{\frac{a+b}{2}x}\cdot\frac{\sin\frac{b-a}{2}x}{\frac{b-a}{2}x}\times\frac{\frac{a+b}{2}x\times\frac{b-a}{2}x}{x^2}$$

$$=\frac{b^2-a^2}{2}\left(\frac{\sin\frac{a+b}{2}x}{\frac{a+b}{2}x}\right)\times\left(\frac{\sin\frac{b-a}{2}x}{\frac{b-a}{2}x}\right)$$

$$\therefore \lim_{x\to 0} f(x)=\frac{b^2-a^2}{2}\times \text{Lt}(\ '')\times \text{Lt}(\ '')$$

$$=\frac{b^2-a^2}{2}\times 1\times 1=\frac{b^2-a^2}{2}.$$

For the continuity of $f(x)$ at $x=0$, we must have

$$\lim_{x\to 0} f(x)=f(0)$$

and therefore for continuity we must define $f(0)=\dfrac{b^2-a^2}{2}$.

Ex.3. Find the points of continuity of the function defined as follows :

$$f(x)=\begin{cases}\dfrac{x^2}{a}-a, & 0<x<a\\ 0, & x=a\\ a-\dfrac{a^3}{x^2}, & x>a\end{cases}$$ [B.U. 53H; Bhag 89; M.U. 93]

Soln. From the given function, it seems that the only point of singularity if that exists, is the point $x=a$. Hence we consider the continuity of the function at the point $x=a$.

Here $f(a-0)=\lim_{x\to a-0}\left[\dfrac{x^2}{a}-a\right]$

$$=\lim_{h\to 0}\left[\frac{(a-h)^2}{a}-a\right]=\frac{a^2}{a}-a=a-a=0;$$

$$\text{and } f(a+0) = \lim_{x \to a+0}\left[a - \frac{a^3}{x^2}\right]$$

$$= \lim_{h \to 0}\left[a - \frac{a^3}{(a+h)^2}\right] = a - \frac{a^3}{a^2} = a - a = 0.$$

Also $f(a) = 0$.

Therefore $f(a+0) = f(a-0) = f(a)$.

Hence the function is continuous at $x = a$.

Evidently the function is continuous for all other values of x.

Hence the function $f(x)$ is continuous for all real values of x.

Ex.4. Show that the function $\phi(x)$ defined as

$$\phi(x) = \begin{cases} 0 \text{ when } x = 0 \\ \frac{1}{2} - x \text{ when } 0 < x < \frac{1}{2} \\ \frac{1}{2} \text{ when } x = \frac{1}{2} \\ \frac{3}{2} - x \text{ when } \frac{1}{2} < x < 1 \\ 1 \text{ when } x = 1 \end{cases}$$

has three points of discontinuity. Find them.

[P.U. 55H, 62H, 65H; Bhag. 93H, 97H]

Soln. We test the continuity of the function at the points $x = 0, \frac{1}{2}$ and 1.

For $x = 0$, we have

$$\phi(0) = 0$$

$$\phi(0+0) = \lim_{x \to 0+0}\left[\frac{1}{2} - x\right] = \lim_{h \to 0}\left[\frac{1}{2} - (0+h)\right] = \frac{1}{2}.$$

Since $\phi(0) \neq \phi(0+0)$, therefore the function is discontinuous at $x = 0$.

For $x = \frac{1}{2}$, we have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$

$$\phi\left(\frac{1}{2}+0\right) = \lim_{x \to \frac{1}{2}+0}\left[\frac{3}{2} - x\right] = \lim_{h \to 0}\left[\frac{3}{2} - \left(\frac{1}{2}+h\right)\right] = 1$$

$$\phi\left(\frac{1}{2}-0\right) = \lim_{x \to \frac{1}{2}-0}\left[\frac{1}{2} - x\right] = \lim_{h \to 0}\left[\frac{1}{2} - \left(\frac{1}{2}-h\right)\right] = 0.$$

Since $\phi\left(\frac{1}{2}-0\right) \neq \phi\left(\frac{1}{2}\right) \neq \phi\left(\frac{1}{2}+0\right)$, therefore the function is discontinuous at $x=\frac{1}{2}$.

Finally, for $x=1$, we have $\phi(1)=1$

$$\phi(1-0)=\underset{x \to 1-0}{\text{Lt}}\left[\frac{3}{2}-x\right]=\underset{h \to 0}{\text{Lt}}\left[\frac{3}{2}-(1-h)\right]=\frac{1}{2}.$$

Since $\phi(1-0) \neq \phi(1)$, therefore the function is discontinuous at $x=1$.

Hence the function $\phi(x)$ is discontinuous at $x=0, \frac{1}{2}$ and 1.

If we draw the graph of the function from $x=0$ to $x=1$, we find that the graph consists of the following :

$y=0$ for $x=0$

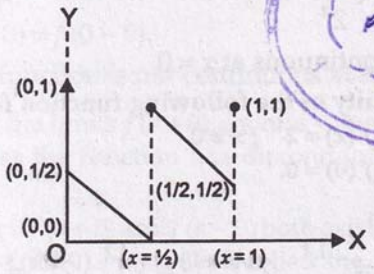
$y=\frac{1}{2}-x$ for $0<x<\frac{1}{2}$ and this is a straight line

$y=\frac{1}{2}$ for $x=\frac{1}{2}$

$y=\frac{3}{2}-x$ for $\frac{1}{2}<x<1$; this is a straight line

$y=1$ for $x=1$.

The graph is shown below :



From the graph it is clear that the function is discontinuous at $x=0, \frac{1}{2}$ and 1

Ex.5. Find out the points of discontinuity of the function

$$f(x)=\frac{1}{2^n} \text{ for } \frac{1}{2^{n+1}}<x \leq \frac{1}{2^n};\ n=0, 1, 2, \ldots \text{ and } f(0)=0.$$

[M.U. 63H]

Soln. From the given function, we have

$$f(x)=\frac{1}{2^n} \text{ for } \frac{1}{2^{n+1}}<x \leq \frac{1}{2^n},$$

$$f(x) = \frac{1}{2^{n-1}} \text{ for } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}.$$

We consider the point $x = \frac{1}{2^n}$.

We have $f\left(\frac{1}{2^n}\right) = \frac{1}{2^n}$,

$$f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}},$$

$$f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}.$$

Since $f\left(\frac{1}{2^n} - 0\right) \neq f\left(\frac{1}{2^n} + 0\right)$, therefore the function $f(x)$ is not continuous at $x = \frac{1}{2^n}$, when $n = 0, 1, 2, 3, 4, \ldots$

For $x = 0$, we see that given $\varepsilon > 0$, we can take m such that $\frac{1}{2^m} < \varepsilon$.

Now $|f(x) - f(0)| = \left|\frac{1}{2^n} - 0\right| < \varepsilon$ for all $n \geq m$

i.e. for all $0 \leq x < \frac{1}{2^n}$.

Hence $f(x)$ is continuous at $x = 0$.

Ex.6. Test the continuity of the following function for $x = 0$;

$$f(x) = 2^{1/x};\ x \neq 0$$
$$f(0) = 0.$$

Soln. Here $f(x) = 2^{1/x}$.

$$\therefore \quad f(0+0) = \lim_{h \to 0} 2^{1/(0+h)} = \lim_{h \to 0} 2^{1/h} = 2^{\infty} = \infty;$$

$$f(0-0) = \lim_{h \to 0} 2^{1/(0-h)} = \lim_{h \to 0} 2^{-1/h} = 2^{-\infty} = 0.$$

Also $f(0) = 0$.

Since $f(0+0) \neq f(0-0)$, therefore the function is not continuous at $x = 0$.

Note A function f is said to have an infinite discontinuity at a point $x = a$ if one or both of the limits $f(a+0)$ and $f(a-0)$ are infinite.

The above function has an infinite discontinuity at $x = 0$.

Ex.7. Discuss the continuity of the following functions at $x = 0$.

(i) $f(x) = \dfrac{1}{1 - e^{1/x}}; x \neq 0$

$f(0) = 0.$ [P.U. 58H, 61H]

(ii) $f(x) = e^{-1/x^2}; x \neq 0$

$f(0) = 0.$ [M.U. 67H, 71H; P.U. 71H]

(iii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}; x \neq 0$

$f(0) = 0.$

Soln. Here $f(x) = \dfrac{1}{1 - e^{1/x}}$.

$$\therefore \quad f(0+0) = \lim_{h \to 0} \frac{1}{1 - e^{1/(0+h)}} = \lim_{h \to 0} \frac{1}{1 - e^{1/h}}$$

$$= \frac{1}{1 - e^{\infty}} = \frac{1}{1 - \infty} = 0.$$

and $$f(0-0) = \lim_{h \to 0} \frac{1}{1 - e^{1/(0-h)}} = \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$$

$$= \frac{1}{1 - e^{-\infty}} = \frac{1}{1 - 0} = 1.$$

Also $f(0) = 0$

Thus $f(0+0) \neq f(0-0)$.

Hence the function is not continuous at the origin.

Note (a) If both the limits $f(a+0)$ and $f(a-0)$ exist but are not equal, we say that the function has discontinuity of the first kind at $x = a$.

(b) Further if $f(a+0)$ and $f(a-0)$ both exist and are unequal, their difference $f(a+0) - f(a-0)$ is called the jump in the function. Thus the above function has a jump of one unit at $x = 0$.

A function having a finite number of jumps in a given interval is called *piece-wise continuous* or *sectionally continuous.*

(ii) Here $f(x) = e^{-1/x^2}$.

$$\therefore \quad f(0+0) = \lim_{h \to 0} e^{-1/(0+h)^2} = \lim_{h \to 0} e^{-1/h^2} = e^{-\infty} = 0.$$

and $f(0-0) = \lim_{h \to 0} e^{-1/(0-h)^2} = \lim_{h \to 0} e^{-1/h^2} = 0$ as before.

Also $f(0) = 0$.

$\therefore\ f(0-0)=f(0)=f(0+0).$

Hence the given function is continuous at $x=0$.

(iii) Here $f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$.

$$\therefore\quad f(0+0)=\lim_{h\to 0}\frac{e^{1/(0+h)}-1}{e^{1/(0+h)}+1}=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$$

$$=\lim_{h\to 0}\frac{1-e^{-1/h}}{1+e^{-1/h}}=\frac{1-e^{-\infty}}{1+e^{-\infty}}$$

$$=\frac{1-0}{1+0}=1.$$

$$\text{and } f(0-0)=\lim_{h\to 0}\frac{e^{1/(0-h)}-1}{e^{1/(0-h)}+1}=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$=\frac{e^{-\infty}-1}{e^{-\infty}+1}=\frac{0-1}{0+1}=-1.$$

Since $f(0+0)\neq f(0-0)$, the function is discontinuous at $x=0$.

Ex.8. Test the following functions for continuity :

(i) $f(x)=\sin\dfrac{1}{x},\ x\neq 0$

$f(0)=0.$

(ii) $f(x)=x\sin\dfrac{1}{x},\ x\neq 0$

$f(0)=0.$ **[M.U. 83H, 98H; R.U. 85; Bhag. 90H, 94]**

(iii) $f(x)=x^2\sin\dfrac{1}{x},\ x\neq 0$

$f(0)=0.$

Soln. (i) We know that as $x\to 0$, the value of $\sin\dfrac{1}{x}$ oscillates between +1 and −1, passing through 0 and intermediate values, an infinite number of times.

Evidently there is no definite number to which $\sin\dfrac{1}{x}$ tends as $x\to 0$ from the right. So the limit on the right does not exist.

Similarly the limit on the left does not exist.

Hence the limit of $\sin\dfrac{1}{x}$ as $x\to 0$ does not exist.

Hence the function is not continuous at $x=0$.

Note If $f(a+0)$ and $f(a-0)$ both do not exist, then f is said to have a discontinuity of the second kind at $x=a$.

If however one of these limits exist but the other does not exist, $\lim_{h \to 0}$ then f is said to have a mixed discontinuity at $x = 0$.

(ii) For any $\varepsilon > 0$, we can take a δ such that $\varepsilon > \delta > 0$.

Now $|f(x) - f(0)| = \left| x \sin \frac{1}{x} - 0 \right|$

$\leq |x|$, since $\left| \sin \frac{1}{x} \right| \leq 1$

and this $< \delta$ provided $|x - 0| < \delta$.

Thus $|f(x) - f(0)| < \varepsilon$, for $\delta < \varepsilon$ for all x such that $|x - 0| < \delta$.

Therefore the function is continuous at $x = 0$.

(iii) Here $|f(x) - f(0)| = \left| x^2 \sin \frac{1}{x} - 0 \right|$

$\leq |x^2| = x^2$, since $\left| \sin \frac{1}{x} \right| \leq 1$.

Now $|x - 0| < \delta \Rightarrow |x| < \delta \Rightarrow |x^2| < \delta^2$.

For any $\varepsilon > 0$, we can choose a δ such that $\varepsilon > \delta^2 > 0$ so that $|f(x) - f(0)| < \delta^2 < \varepsilon$ for all x such that $|x - 0| < \delta$.

Therefore the function is continuous at $x = 0$.

Ex.9. Let a function f be defined by $f(x) = x - [x]$ where x is a positive variable and $[x]$ denotes the integral part of x. Show that this function is discontinuous for all integral values of x and continuous for all others. **[P.U. 59H; Bhag 96H; Haz. 2004H]**

Soln. From the definition of the integral function, it follows that

$[x] = n$ if $n \leq x < n + 1$ and $[x] = (n - 1)$ if $n - 1 \leq x < n$.

Hence $f(x) = x - (n - 1)$ if $n - 1 \leq x < n$

$f(x) = 0$ if $x = n$

$f(x) = x - n$ if $n < x < n + 1$,

where n is an integer.

We test the continuity of the function at the point $x = n$.

We have, $f(n - 0) = \lim_{x \to n - 0} [x - (n - 1)]$

$= \lim_{h \to 0} [(n - h) - (n - 1)]$

$= \lim_{h \to 0} (-h + 1) = 1.$

and $f(n + 0) = \lim_{x \to n + 0} [x - n] = \lim_{h \to 0} [n + h - n]$

Also $f(n) = 0$.

Since $f(n-0) \neq f(n+0)$, therefore the function $f(x)$ is discontinuous at $x = n$. That is, the function $f(x)$ is discontinuous for all integral values of x.

Obviously it is continuous for all other values. This can be shown as follows :

Let x be not an integer.

Then there ar two integers n and $n+1$ such that $n < x < n+1$.

We can always choose a δ such that

$$n < x - \delta < x < x + \delta < n + 1.$$

If $|h| < \delta$, then we have

$$f(x+h) - f(x) = \{(x+h) - [x+h]\} - \{x - [x]\}$$
$$= (x + h - n) - (x - n) = h \to 0 \text{ as } h \to 0.$$

Hence $f(x)$ is continuous at non-integral values of x.

Graph : Given $f(x) = 0$ if $x = n$ and $f(x) = x - n$ if $n < x < n+1$.

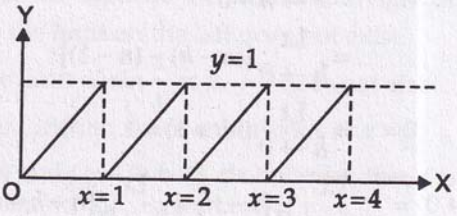
Therefore putting $n = 1, 2, 3, \ldots$ successively, we see that the graph consists of the following :

$y = x$ when $0 < x < 1$
$y = 0$ when $x = 1$
$y = x - 1$ when $1 < x < 2$
$y = 0$ when $x = 2$
$y = x - 2$ when $2 < x < 3$
$y = 0$ when $x = 3$
$y = x - 3$ when $3 < x < 4$
$y = 0$ when $x = 4$
and so on.

The graph is shown below by thick lines from $x = 0$ to $x = 4$.

Remark : From the graph, the following facts are evident.

(i) The function is discontinuous for all integral values of x i.e.



at the points $x = 1, 2, 3, \ldots$ but it is continuous for all others.

(ii) The function is bounded between 0 and 1. The lower bound zero is attained but the upper bound 1 is not attained, since $f(x) \neq 1$ for any value of x whatsoever.

1.20 Theorem

The necessary and sufficient condition for a function $f(x)$ defined on $I \subset R$ to be continuous at $a \in I$ is that for each sequence $\{a_n\}$ which converges to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ i.e. } \lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right). \quad \text{[M.U. 72H]}$$

Proof : The condition is necessary.

Let $f(x)$ be continuous at $a \in I$.

Then given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \quad \ldots(1)$$

Again, let $\{a_n\}$ be a sequence such that $\lim_{n \to \infty} a_n = a$.

Therefore there exists a positive integer m such that

$$n > m \Rightarrow |a_n - a| < \delta. \quad \ldots(2)$$

We want to show that $\lim_{n \to \infty} f(a_n) = f(a)$.

Setting $x = a_n$ in (1), we get $|a_n - a| < \delta \Rightarrow |f(a_n) - f(a)| < \varepsilon \ldots(3)$

Thus combining (2) and (3), we get

$$n > m \Rightarrow |f(a_n) - f(a)| < \varepsilon.$$

Hence $\lim_{n \to \infty} f(a_n) = f(a)$.

The condition is sufficient : We suppose that for every sequence $\{a_n\}$ which converges to a, we have $\lim_{n \to \infty} f(a_n) = f(a)$.

Then we want to show that $f(x)$ is continuous at $x = a$.

On the contrary, suppose that $f(x)$ is not continuous at $x = a$.

This means that there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there is an x such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

If we take $\delta = \frac{1}{n}$, we see that for each positive integer n, there exists a a_n (setting $x = a_n$) such that $|a_n - a| < \frac{1}{n}$ but $|f(a_n) - f(a)| \geq \varepsilon$.

This means that $\underset{n \to \infty}{\text{Lt}}\, a_n = a$ but $\underset{n \to \infty}{\text{Lt}}\, f(a_n) \neq f(a)$.

This contradicts our hypothesis that $\underset{n \to \infty}{\text{Lt}}\, f(a_n) = f(a)$.

Hence $f(x)$ must be continuous at $x = a$.

Ex.10. Show that the function f defined on R by

$f(x) = 0$ **when** x **is rational**

$= 1$ **when** x **is irrational**

is discontinuous for every $x \in R$.

Soln. Let a be any rational number and let ε be a positive number less than 1.

Let us choose any $nbd(a - \delta, a + \delta)$. Now in this neighbourhood there exists an infinity of both rational as well as irrational numbers.

Consider any irrational number $a + h$ in $(a - \delta, a + \delta)$.

Then, we have $f(a + h) = 1$ and $f(a) = 0$.

Now, we have $|f(a + h) - f(a)| = |1 - 0| = 1$.

Thus $|f(a + h) - f(a)| \not< \varepsilon$ (because $0 < \varepsilon < 1$).

Hence $f(x)$ is discontinuous at $x = a$. But a is any arbitrary rational number.

Hence $f(x)$ is discontinuous for every rational number.

Similarly it can be shown that $f(x)$ is discontinuous for every irrational number.

Hence $f(x)$ is discontinuous for every $x \in R$.

Second method : Let a be a rational number.

Let us take a sequence $\{a_n\}$ of rational numbers such that

$$\underset{n \to \infty}{\text{Lt}}\, a_n = a.$$

Now, from the definition of the function, we have $f(a_n) = 0$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, f(a_n) = 0 \text{ where } \underset{n \to \infty}{\text{Lt}}\, a_n = a.$$

Next, let us consider a sequence $\{b_n\}$ of irrational numbers such that $\underset{n \to \infty}{\text{Lt}}\, b_n = a$.

In this case, $f(b_n) = 1$

$\therefore \underset{n \to \infty}{\text{Lt}} f(b_n) = 1$ where $\underset{n \to \infty}{\text{Lt}}\, b_n = a$.

Thus we get $\underset{n \to \infty}{\text{Lt}} f(x_n) = 0$ or 1 according as $\{x_n\}$ denotes a sequence of rational or irrational numbers converging in either case to a. Hence we conclude that $\underset{n \to \infty}{\text{Lt}} f(x_n)$ does not exist.

Consequently $f(x)$ is not continuous at $x = a$ where a is a rational number.

Similarly it can be shown that $f(x)$ is not continuous at $x = a$ where a is an irrational number.

Thus $f(x)$ is discontinuous for every $x \in R$.

Ex.11. Show that the function f defined on R by

$f(x) = 1$ when x is rational

$f(x) = -1$ when x is irrational

is discontinuous for every $x \in R$.

Soln. Firstly suppose that x is a rational number.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we choose an irrational number x_n such that $|x_n - x| < \frac{1}{n}$.

Then $\{x_n\}$ converges to x.

Now $f(x_n) = -1$ for all n since x_n is irrational.

$\therefore \underset{n \to \infty}{\text{Lt}} f(x_n) = -1$

But $f(\text{Lt}\, x_n) = f(x) = 1$

Therefore $\underset{n \to \infty}{\text{Lt}} f(x_n) \neq f(x)$.

Hence f is discontinuous at each rational point.

Next, let x be irrational number.

For each positive integer n we choose a rational number x_n such that $|x_n - x| < \frac{1}{n}$.

Then $\{x_n\}$ converges to x.

Now $f(x_n) = 1$ for all n, since x_n is a rational number.

$\therefore \quad \text{Lt}\, f(x_n) = 1.$

But $f(\text{Lt}\, x_n) = f(x) = -1.$

Therefore $\underset{n \to \infty}{\text{Lt}} f(x_n) \neq f(x).$

Hence f is discontinuous at each irrational point.

Thus f is discontinuous for all $x \in R$, i.e. it is totally discontinuous.

This function is known as **Dirichlet's function.**

Ex.12. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} x \text{ when } x \text{ is irrational} \\ -x \text{ when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

Soln. First let $a \neq 0$ be any rational number so that $f(a) = -a$.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that $|a_n - a| < \frac{1}{n}$.

Thus the sequence $\{a_n\}$ converges to a.

But $\underset{n \to \infty}{\text{Lt}} f(a_n) = \underset{n \to \infty}{\text{Lt}} (a_n) = a.$

Thus $\underset{n \to \infty}{\text{Lt}} f(a_n) \neq f(a)$ when $a \neq 0$.

Hence the function is discontinuous at any rational number other than zero.

Similarly it can be shown that the function is discontinuous at every irrational point.

That the function is continuous at $x = 0$ can be shown as follows :

Let $\varepsilon > 0$ be given and let $\delta \leq \varepsilon$.

Then $|x - 0| < \delta = |x| < \delta$

$\Rightarrow |f(x) - f(0)| = |-x - 0| = |-x| = |x| < \varepsilon$ when x is rational

and $|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon$ when x is irrational

Thus $|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$.

Hence the function is continuous at $x = 0$.

Ex.13. **Let $\phi(x) = x$ when x is rational**
$= 1 - x$ when x is irrational.
Show that $\phi(x)$ assumes every value between 0 and 1 once and once only as x increases from 0 to 1, but is discontinuous for every value of x except $x = \frac{1}{2}$. **[Bhag. 75H]**

Soln. Let a be any arbitrary value between 0 and 1.

We need to show that $\phi(x) = a$ for one and only one $x \in [0, 1]$.

First assume that a is rational. Then its preimage point will be determined from the first equation $\phi(x) = x$.

Obviously $\phi(a) = a$, i.e. at $x = a$, $\phi(x) = a$ where $0 \le a \le 1$.

Further, suppose that there is another point x' such that

$$\phi(x') = a = \phi(a).$$

Then x' must be rational and $x' = a$.

Hence if a is rational, $\phi(x) = a$ for one and only one value of x in $[0, 1]$, namely $x = a$.

Next, suppose that a is irrational.

Then $1 - a$ is also irrational. The question is what is the value of x such that $\phi(x) = a$ (an irrational number).

The answer is provided by the equation $\phi(x) = 1 - x$.

Putting $x = 1 - a$ in this equation we get

$$\phi(1 - a) = 1 - (1 - a) = a.$$

Hence there is a value of $x = 1 - a$ for which $\phi(x) = a$.

Moreover we cannot find another x say x' such that $\phi(x') = a$.

For, $\phi(x') = 1 - x'$ giving $1 - x' = a$ i.e. $x' = 1 - a$.

Hence if a is irrational, $\phi(x) = a$ for only one value of x namely $x = 1 - a \in [0, 1]$.

Further suppose that x is continuous at $x = t$.

Hence the limit of $\phi(x)$ as $x \to t$ through rational values is $\phi(t)$ while that through irrational values is also $\phi(t)$.

But Lt $\phi(t) = t$ when $x \to t$ through rational values

and Lt $\phi(t) = 1 - t$ when $x \to t$ through irrational values.

Therefore $t = 1 - t \Rightarrow 2t = 1 \Rightarrow t = \frac{1}{2}$.

Hence $\phi(x)$ is discontinuous for every value of x except $x = \frac{1}{2}$.

Ex.14. Show that if $f(x)$ is continuous in the closed interval [0, 1] and vanishes for rational x, then it vanishes everywhere.

[P.U. 71H, 73H; B.U. 57H; Bhag. 70H, 96H; M.U. 68H, 71H, 73H, 96H; Dumka 97H]

Soln. It suffices to prove that $f(x) = 0$ when x is any irrational.

Let α be any irrational $\in [0, 1]$.

We need to show that $f(\alpha) = 0$.

We express α as an infinite continued fraction and let its nth convergent be r_n. Then $\{r_n\}$ is a sequence of rational numbers whose limit is α.

Since $f(x)$ is continuous at $x = \alpha$, therefore $\underset{n \to \infty}{\text{Lt}} f(x) = f(\alpha)$ for all modes of approach of x to α.

Let x approach α through the sequence $\{r_n\}$.

Hence $\underset{n \to \infty}{\text{Lt}} f(r_n) = f(\text{Lt } r_n) = f(\alpha)$.

But $f(r_n) = 0$ for all n (given).

Hence $\underset{n \to \infty}{\text{Lt}} f(r_n) = 0$ and consequently $f(\alpha) = 0$.

This completes the proof.

Exercise 1

1. Evaluate the following limits (if they exist)

(i) $\underset{x \to 2}{\text{Lt}} \dfrac{x^2 - 3x + 2}{x - 2}$ (ii) $\underset{x \to 2}{\text{Lt}} \dfrac{x^6 - 64}{x^4 - 16}$

(iii) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos 3x}{x^2}$ (iv) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - 1}{x}$

(v) $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{b^x - 1}$ (vi) $\underset{x \to 0}{\text{Lt}}\, e^{-1/x}$

(vii) $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x} e^{1/x}$ (viii) $\underset{x \to a}{\text{Lt}} \dfrac{1}{1 - e^{1/(x-a)}}$

2. Discuss the existence of $\underset{x \to 0}{\text{Lt}} \sin \dfrac{1}{x}$ and $\underset{x \to 0}{\text{Lt}}\, x \sin \dfrac{1}{x}$.

[P.U. 64H; M.U. 69H, 72H]

Examine if $\underset{x \to 0}{\text{Lt}}\, x \sin \dfrac{1}{x} = \underset{x \to 0}{\text{Lt}}\, x \times \underset{x \to 0}{\text{Lt}} \sin \dfrac{1}{x}$ makes sense.

[P.U. 64H]

[Hint : If $\text{Lt}\, f(x)$ and $\text{Lt}\, g(x)$ exist separately, then $\text{Lt}\, \{f(x)\, g(x)\}$ exists and $= \text{Lt}\, f(x) \times \text{Lt}\, g(x)$]

3. Prove that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$. [R.U. 72H]

4. Examine the continuity of $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} \text{ when } x \neq 1 \\ 2 \text{ when } x = 1 \end{cases}$ [R.U. 73H]

5. Examine for continuity when $f(x) = \dfrac{x^3-1}{x-1}$ for $x \neq 1$
$= 2$ for $x = 1$. [M.U. 71H]

6. Examine the continuity of $f(x)$ at $x = \dfrac{1}{2}$ when

$$f(x) = \begin{cases} x \text{ for } 0 \leq x \leq \dfrac{1}{2} \\ 1 - x \text{ when } \dfrac{1}{2} < x \leq 1 \end{cases}$$ [M.U. 73H, 92]

7. Explain the meaning of continuity and discontinuity of a function at a point. Show that the function $f(x)$ where

$$f(x) = \begin{cases} x \text{ when } 0 \leq x < \dfrac{1}{2} \\ 1 \text{ when } x = \dfrac{1}{2} \\ 1 - x \text{ when } \dfrac{1}{2} < x < 1. \end{cases}$$

is discontinuous for $x = \dfrac{1}{2}$.

What is the nature of the discontinuity? [P.U. 57H; M.U. 80H]

8. Consider the function defined as follows :

$f(x) = x$ when $x < 1$

$f(x) = 2 - x$ when $x \geq 1$.

Is $f(x)$ continuous at $x = 1$? Justify you answer. [P.U. 60H]

9. Explain whether $f(x)$ is continuous at $x = 0$ where

$f(x) = 2 + x$ if $x \geq 0$

$f(x) = 2 - x$ if $x < 0$. [R.U. 63H, 89H]

10. A function $f(x)$ is defined as follows :

$f(x) = 1 + 4x$ when $x < 1$

$f(x) = 9 - 4x$ if $x \geq 1$.

Is $f(x)$ continuous at $x = 1$? [B.U. 66H]

11. Discuss the continuity of the following function at $x = 0$;

$f(x) = 5x^2 + 3x + 7$ when $-1 < x \leq 0$

$f(x) = 5x^2 - 3x + 7$ when $0 < x < 1$ **[M.U. 67H]**

12. Discuss the continuity of $f(x) = \begin{cases} 1 \text{ when } x \text{ is rational} \\ 0 \text{ when } x \text{ is irrational.} \end{cases}$

[Bhag.U. 74H]

[Hint : Between any two real numbers there lie infinity of real numbers.]

13. Examine whether the following function is continuous at $x = 0$.

$$f(x) = \frac{\sin 2x}{x} \text{ when } x \neq 0$$

$f(0) = 1.$

14. Examine the function defined below for continuity at $x = 0$

$$f(x) = \frac{\sin^2 ax}{x^2} \text{ when } x \neq 0$$

$f(0) = 1.$

15. Discuss the continuity of $f(x) = \dfrac{\cos ax - \cos bx}{x^2}; x \neq 0$

$f(0) = 1.$ **[M.U. 63H]**

16. Is the following function continuous at the origin?

$$f(x) = \frac{\cos ax - \cos bx}{x^2}, \text{ when } x \neq 0$$

$$f(0) = \frac{b^2 - a^2}{2}.$$ **[R.U. 64H]**

17. Discuss the continuity of the following functions :

(i) $f(x) = x^3 + 3x$ (ii) $f(x) = x + \dfrac{1}{x}$

(iii) $f(x) = e^{-1/x}$ (iv) $f(x) = \cos \dfrac{1}{x}$

(v) $f(x) = |x|$ **[M.U. 89; R.U. 90, 91H; B.U. 94]**

(vi) $f(x) = \dfrac{\sin x}{x}$ when $x \neq 0$

$f(0) = 1.$

(vii) $f(x) = \dfrac{1-\cos x}{x^2}$ when $x \neq 0$

$f(0) = 1$ [B.U. 90]

(viii) $f(x) = x \sin \dfrac{1}{x}$ when $x \neq 0$

$f(0) = 1$ [Bhag.U. 90H, 92]

(ix) $f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}$ when $x \neq 0$

$f(0) = 0$ [R.U. 95H]

(x) $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ when $x \neq 0$

$f(0) = 1.$ [M.U. 83H]

(xi) $f(x) = (1+x)^{1/x}$ when $x \neq 0$

$f(0) = 1.$

18. If $f(x) = (x-a) \sin \dfrac{1}{x-a}$, find $f(a+0)$ and $f(a-0)$. Is the function continuous at $x = a$?

19. Show that $f(x)$ is discontinuous at $x = 0$ if

$f(x) = 3^{1/x}$ when $x \neq 0$

$= 2$ when $x = 0.$

20. Discuss the continuity of the following function at $x = 1$

$$f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$ [P.U. 63H]

21. Test the continuity of the following function at $x = a$:

$$f(x) = \frac{1}{x-a} \operatorname{cosec} \frac{1}{x-a} \text{ when } x \neq a$$

$f(a) = 0.$

[Here $f(x)$ is discontinuous at $x = a$ having an infinite discontinuity of the second kind.]

22. Discuss the continuity of the following function at $x = 0$

$$f(x) = x^2 \sin \frac{1}{x} + x \cos \frac{1}{x} \text{ when } x \neq 0$$

$f(0) = 0.$ [M.U. 67H]

[It is the sum of two continuous functions.]

23. Let $y = E(x)$ where $E(x)$ denotes the integral part of x. Prove that this function is discontinuous where x has an integral values. Also draw the graph.

[Here $y = E(x) = [x] = n$ for $n \le x < n+1$]

24. Draw the graph of the function y defined by the statement "y is the smallest positive number that makes $x + y$ an integer."

For what values of x is the function discontinuous? **[P.U. 51H]**

[Here $y = n - x$ when $n - 1 \le x < n$

$y = (n+1) - x$ when $n \le x \le n+1$.]

25. Show that a monotonic function over $]a, b[$ cannot have a discontinuity of the second kind.

[Soln. Let f be monotonic increasing function over $]a, b[$.

Let c be any arbitrary point in $]a, b[$.

Then $f(x) \le f(c)$ for all $x \le c$.

Let $L = \sup_{x<c} \{f(x)\}$

Then from the definition of supremum, for any $\varepsilon > 0$, there exists a point $c - \delta (\delta > 0)$ such that $L - \varepsilon < f(c - \delta) \le L$.

Since f is m.i., we have $L - \varepsilon < f(x) \le L$ for $c - \delta < x < c$.

Hence given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ for } c - \delta < x < c.$$

Thus $\underset{x \to c-0}{\text{Lt}} f(x) = L$.

Similarly, $\underset{x \to c+0}{\text{Lt}} f(x)$ exists $= M = \inf_{c<x} \{f(x)\}$

Hence f cannot have a discontinuity of the second kind at c.

But c is any point in $]a, b[$.

Hence f cannot have a discontinuity of the second kind at any point in $]a, b[$.

26. Show that the function f defined by $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$

is discontinuous at every point. **[M.U. 97H]**

27. Show that the function f defined by $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$

is continuous only at $x = 0$.

28. Let f be continuous on $[a, b]$ and suppose that $f(x) = 0$ for every rational x in $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

29. Let a function $f: R \to R$ satisfy the equation $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. If $f(x)$ is continuous at the point $a \in R$, show that it is continuous for all $x \in R$. **[R.U. 94H, 96H]**

[**Hint :** Since $f(x)$ is continuous at $x = a$, we have

$$\lim_{h \to 0} f(a+h) = f(a)$$

$$\Rightarrow \lim_{h \to 0} \{f(a) + f(h)\} = f(a); \text{ from the given functional equation}$$

$$\Rightarrow f(a) + \lim_{h \to 0} f(h) = f(a)$$

$$\Rightarrow \lim_{h \to 0} f(h) = 0 \qquad \ldots(1)$$

Let $x \in R$ be any real number.

Then $\lim_{h \to 0} f(x+h) = \lim_{h \to 0} \{f(x) + f(h)\}$

$$= f(x) + \lim_{h \to 0} f(h) = f(x) + 0; \text{ from (1)}$$

$$= f(x).$$

Hence $f(x)$ is continuous for all x.]

30. If f is continuous on R and satisfies the functional equation $f(x+y) = f(x) + f(y)$ for all $x, y \in R$, prove that $f(x) = ax$ for all $x \in R$ where a is a constant. **[B.U. 2000H]**

[**Hint :** Putting $y = 1$, we have $f(x+1) = f(x) + f(1)$

$\Rightarrow f(x+1) - f(x) = f(1) \Rightarrow f(x) - f(x-1) = f(1)$.

Now put $x = x-1, x-2, \ldots 1$ and then adding we get

$f(x) = xf(1)$.]

31. If f is continuous, not identically zero on R, and satisfies the functional equation $f(xy) = f(x) + f(y)$ for all $x, y \in R$, prove that there is an $a > 0$ $(a \neq 1)$ such that $f(x) = \log_a x$ for all $x \in R^+$.

ANSWERS

1. (i) 1 (ii) 0 (iii) $\frac{9}{2}$ (iv) 1 (v) $\frac{\log a}{\log b}$ (vi) does not exist (vii) does not exist (viii) does not exist

4. continuous at $x = 1$ 5. continuous except at $x = 1$

6. continuous at $x = \frac{1}{2}$ **9.** continuous

10. continuous **11.** continuous **12.** discontinuous

13. discontinuous **14.** discontinuous if $a^2 \neq 1$

15. discontinuous unless $\frac{b^2 - a^2}{2} = 1$. **16.** yes

17. (i) continuous (ii) continuous except at $x = 0$

(iii) discontinuous at $x = 0$

(iv) discontinuous at $x = 0$ and hence for all x

(v) continuous at $x = 0$ (vi) continuous at $x = 0$

(vii) discontinuous at $x = 0$ (viii) discontinuous at $x = 0$

(ix) discontinuous at $x = 0$ (x) discontinuous at $x = 0$

(xi) removable discontinuity

18. each = 0, no **20.** continuous

□□□

CHAPTER

2

Differentiation of a Function

2.1 Introduction

In the previous chapter we have studied a very important concept of Mathematics namely the continuity of a function. In the present chapter we shall study about an equally important concept of Mathematics which is called the operation of differentiation of a function and it will be seen that continuity and differentiation (or derivability) of a function together act as a master key to further mathematical development. In the beginning of this chapter we shall see that the continuity of a function is a necessary but not sufficient condition for the existence of the finite derivative of the function. In other words, if the function $f(x)$ has got a finite derivative for any value of x, then the function is necessarily continuous at that point, but if a function is continuous at a given point, then the function need not necessarily possess a finite derivative at that point.

2.2 Definition of Derivative

Definition : Let $f(x)$ be a function defined on the open interval $I =]a, b[$ and let c be a point $\in I$.

Then whenever the limit $\underset{x \to c}{\text{Lt}} \dfrac{f(x) - f(c)}{x - c}$ ...(1)

or equivalently $\underset{h \to 0}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h}$

exists (finite or infinite), $f(x)$ is said to have a derivative at $x = c$ and is denoted by $f'(c)$.

Thus $f'(c) = \underset{h \to 0}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h}$.

The another name for derivative is differential coefficient.

It should be noted that in order to examine the limit (1) we have to consider the following two limits corresponding to right hand limit and left hand limit of a function viz.

$$\lim_{x \to c+0} \frac{f(x)-f(c)}{x-c} \text{ and } \lim_{x \to c-0} \frac{f(x)-f(c)}{x-c}$$

or equivalently $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}; h>0$ ...(2)

and $\lim_{h \to 0} \frac{f(c-h)-f(c)}{-h}; h>0.$ ...(3)

Since (2) and (3) are respectively the right hand limit and left hand limit of the function $\frac{f(x)-f(c)}{x-c}$ when $x \to c$, therefore (2) and (3) may be conveniently denoted by $Rf'(c)$ and $Lf'(c)$ respectively., The symbols $Rf'(c)$ and $Lf'(c)$ are used to mean respectively the right hand and left hand derivatives of $f(x)$ at $x=c$.

The function $f(x)$ is said to be differentiable at $x=c$ if $Rf'(c)$ and $Lf'(c)$ both exist and are equal.

Thus for the existence of $f'(c)$ we must have :

(i) $Rf'(c)$ should exist

(ii) $Lf'(c)$ should exist

(iii) $Rf'(c) = Lf'(c)$.

Note We shall use the symbol $Df(x)$ whenever necessary to mean the derivative of $f(x)$.

2.3 Derivability on the Closed Interval [a, b]

When $f(x)$ is defined in the open interval $]a, b[$, $f(x)$ is said to be derivable in the open interval $]a, b[$ if it is derivable at every point of $]a, b[$. In such a case as we know, the derivative $f'(c)$ where $a<c<b$ is defined by the limit $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$.

On the other hand if $f(x)$ is defined in the closed interval $[a, b]$, then $f(x)$ is said to be derivable in $[a, b]$ if

(i) $f'(c)$ exists for every $c \in]a, b[$

i.e. for every c such that $a<c<b$.

(ii) $Rf'(a)$ exists

i.e. $f(x)$ is derivable from the right at $x=a$.

(iii) $Lf'(b)$ exists

i.e. $f(x)$ is derivable from the left at $x = b$.

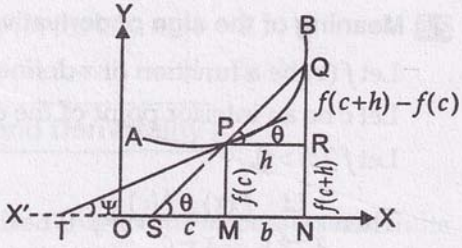
Thus if $f(x)$ is derivable on $]a, b[$ and also at the end-points a and b, then we say that it is derivable on the closed interval $[a, b]$.

Remark : Let f be a function defined on the interval I and let c be an interior point of I. From the definition of $f'(c)$, assuming it to exist, for a given $\varepsilon > 0$, there corresponds a number $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

2.4 Geometrical Meaning of Derivative

Let $APQB$ be the graph of the function $y = f(x)$ defined in the interval $[a, b]$.



Let P be any point on the curve and let Q be any other point on the curve very close to P. From P and Q draw PM and QN perpendiculars respectively on OX.

Let $OM = c$, then $ON = c + h$.

Therefore, we have $PM = f(c)$ and $QN = f(c + h)$.

From P draw $PR \perp QN$.

$\therefore \quad PR = MN = (c + h) - c = h$

and $QR = QN - RN = QN - PM = f(c + h) - f(c)$.

Join QP and produce it to meet OX in S such that $\angle PSX = \theta$.

$\therefore \quad \angle QPR = \theta$.

Hence from the ΔQPR, $\tan\theta = \dfrac{QR}{PR} = \dfrac{f(c + h) - f(c)}{h}$ ...(1)

Let the tangent at P meet the x-axis in T such that $\angle PTX = \psi$.

Now let the point Q move along the curve to coincide with the point P so that when $Q \to P$, $h \to 0$ and the chord QP approaches the tangent line PT as its limiting position and consequently $\theta \to \psi$.

Therefore on taking limits, the equation (1) gives

$$\lim_{\theta \to \psi} \tan\theta = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

or $\quad \tan\psi = f'(c)$.

Thus $f'(c)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P(c, f(c))$ makes with the x-axis.

Obs. : We recollect here certain results regarding differentiation of sum or product of two functions which we have done in previous year. They are : If $f(x)$ and $g(x)$ are two functions such that $f'(x)$ and $g'(x)$ exist and are finite, then

(i) $D\{f(x) \pm g(x)\} = Df(x) \pm Dg(x) = f'(x) \pm g'(x)$

(ii) $D\{f(x) \cdot g(x)\} = f(x)\, g'(x) + g(x) \cdot f'(x)$

(iii) $D\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)\, g(x) - g'(x)\, f(x)}{\{g(x)\}^2}$ provided $g(x) \neq 0$.

2.5 Meaning of the sign of derivative

Let $f(x)$ be a function of x defined over the interval $I = [a, b]$.

Let c be an interior point of the domain of the function f.

Let $f'(c) > 0$.

Then $\underset{x \to c}{\text{Lt}} \dfrac{f(x) - f(c)}{x - c} = f'(c) > 0$.

This means that if $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

That is, $x \in\ ]c - \delta, c + \delta[$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} \in\ ]f'(c) - \varepsilon, f'(c) + \varepsilon\ [\qquad \ldots(1)$$

We take $\varepsilon < f'(c)$ so that from (1) we conclude that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ whenever } x \in\ ]c - \delta, c + \delta[;\ x \neq c$$

We then have $f(x) - f(c) > 0$ whenever $x > c$ and $x \in\ ]c - \delta, c + \delta[$,

and $\quad f(x) - f(c) < 0$ whenever $x < c$ and $x \in\ ]c - \delta, c + \delta[$.

This means that $f(x) > f(c)$ whenever $x \in\ ]c, c + \delta[$

and $\quad f(x) < f(c)$ whenever $x \in\ ]c - \delta, c[$.

Thus we have shown that if $f'(c) > 0$, there exists a nbd. $]c - \delta, c + \delta[$ such that

(i) $f(x) > f(c)$ for all $x \in\ ]c, c + \delta[$

and (ii) $f(x) < f(c)$ for all $x \in\ ]c - \delta, c[$.

From (i) and (ii) we see that $f(x)$ is increasing at $x = c$.

If $f'(c) < 0$, it can similarly be shown that there exists a nbd. $]c-\delta, c+\delta[$ of c such that

(iii) $f(x) > f(c)$ for all $x \in]c-\delta, c[$

and (iv) $f(x) < f(c)$ for all $x \in]c, c+\delta[$.

Thus if $f'(c) < 0$, then the function is decreasing at $x = c$.

For the end-points a and b it can be shown that there exist nbds. $]a, a+\delta[$ of a and $]b-\delta, b[$ of b such that

$f'(a) > 0 \Rightarrow f(x) > f(a)$ for all $x \in [a, a+\delta[$

$f'(a) < 0 \Rightarrow f(x) < f(a)$ for all $x \in [a, a+\delta[$

$f'(b) > 0 \Rightarrow f(x) > f(b)$ for all $x \in]b-\delta, b]$

$f'(b) < 0 \Rightarrow f(x) > f(b)$ for all $x \in]b-\delta, b]$

Continuity and Derivability

2.6 Theorem

If a function defined in a closed interval $[a, b]$ possesses a finite derivative at a point of the interval, then $f(x)$ is also continuous at that point. **[P.U. 98H, 2000H; Bhag. 94H; 97H, 2000H, 2002H; M.U. 96H, 98H, 2004H; B.U. 2001H, 2004H]**

Proof : Let $f(x)$ have a finite derivative $f'(c)$ at $x = c$ where $c \in [a, b]$.

Then $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$.

Now we observe that $f(c+h) - f(c) = \frac{f(c+h)-f(c)}{h} \cdot h$

$$\therefore \lim_{h \to 0} \{f(c+h) - f(c)\} = \lim_{h \to 0} \left\{\frac{f(c+h)-f(c)}{h} \cdot h\right\}$$

$$= \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \times \lim_{h \to 0} h$$

$$= f'(c) \times 0$$

$$= 0; \because f'(c) \text{ is finite.}$$

Thus we have $\lim_{h \to 0} f(c+h) = f(c)$

which $\Rightarrow$ that $f(x)$ is continuous at $x = c$ where $c \in [a, b]$.

Note The above theorem provides us with a necessary condition for the existence of a finite derivative viz. the necessary condition for a function to possess a finite derivative at a point is that it is continuous at that point. But the converse of the above theorem

is not necessarily true.

In other words, if a function is continuous at a point, then it may or may not possess a derivative at that point. [M.U. 91H]

Ex. Let a function $f(x)$ be defined in [0, 1] such that

$$f(x) = x \text{ when } 0 \le x \le \frac{1}{2}$$

$$= 1 - x \text{ when } \frac{1}{2} < x \le 1.$$ [M.U. 92]

Soln. We will see that this function is continuous at $x = \frac{1}{2}$ but is not differentiable at $x = \frac{1}{2}$.

For continuity of $f(x)$ at $x = \frac{1}{2}$,

$$f\left(\frac{1}{2} - 0\right) = \underset{h \to 0}{\text{Lt}} f\left(\frac{1}{2} - h\right) = \underset{h \to 0}{\text{Lt}} \left(\frac{1}{2} - h\right) = \frac{1}{2},$$

$$f\left(\frac{1}{2} + 0\right) = \underset{h \to 0}{\text{Lt}} f\left(\frac{1}{2} + h\right) = \underset{h \to 0}{\text{Lt}} \left\{1 - \left(\frac{1}{2} + h\right)\right\} = \frac{1}{2}.$$

Also $f\left(\frac{1}{2}\right) = \frac{1}{2}$

$\therefore \quad f\left(\frac{1}{2} + 0\right) = f\left(\frac{1}{2} - 0\right) = \frac{1}{2}.$

Hence $f(x)$ is continuous at $x = \frac{1}{2}$.

For derivability of $f(x)$ at $x = \frac{1}{2}$,

$$Lf'\left(\frac{1}{2}\right) = \underset{h \to 0}{\text{Lt}} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h} = \underset{h \to 0}{\text{Lt}} \frac{\left(\frac{1}{2} - h\right) - \frac{1}{2}}{-h} = +1$$

$$Rf'\left(\frac{1}{2}\right) = \underset{h \to 0}{\text{Lt}} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h} = \underset{h \to 0}{\text{Lt}} \frac{1 - \left(\frac{1}{2} + h\right) - \frac{1}{2}}{h} = -1.$$

$\therefore \quad Lf'\left(\frac{1}{2}\right) \neq Rf'\left(\frac{1}{2}\right)$

$\therefore \quad f'\left(\frac{1}{2}\right)$ does not exist.

Remark : The continuity and differentiability can be immediately visualised by drawing the graph of the function.

Note If a function possesses an infinite derivative at a point then it may or may not be continuous at that point.

This will be illustrated by means of two examples :

Ex.1. Let a function $f(x)$ be defined as follows :

$$f(x) = +\sqrt{x},\ x \geq 0$$
$$= -\sqrt{x},\ x < 0.$$

Soln. We will show that $f(x)$ possesses a derivative at $x = 0$, derivative being $+\infty$; also $f(x)$ is continuous at $x = 0$.

$$\text{Here } Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sqrt{h} - 0}{-h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = +\infty,$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h} - 0}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = +\infty.$$

Hence $f'(0) = +\infty$.

Next, we have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \sqrt{h} = 0,$$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \sqrt{h} = 0.$$

Also $f(0) = 0$.

Thus $f(x)$ is continuous at $x = 0$.

Ex.2. Let a function $f(x)$ be defined in [0, 1] such that

$$f(x) = 0 \text{ when } 0 \leq x < \frac{1}{2}$$
$$= 1 \text{ when } x = \frac{1}{2}$$
$$= 2 \text{ when } \frac{1}{2} < x \leq 1.$$

Soln. We will show that $f(x)$ possesses a derivative at $x = \frac{1}{2}$, the derivative being $+\infty$, but $f(x)$ is discontinuous at $x = \frac{1}{2}$.

$$\text{Here } Lf'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{0-1}{-h} = \lim_{h\to 0} \frac{1}{h} = +\infty,$$

$$Rf'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2-1}{h} = \lim_{h\to 0} \frac{1}{h} = +\infty.$$

$$\therefore \quad f'\left(\frac{1}{2}\right) = +\infty.$$

$$\text{Next we have } f\left(\frac{1}{2}+0\right) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right) = \lim_{h\to 0} (2) = 2,$$

$$f\left(\frac{1}{2}-0\right) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right) = \lim_{h\to 0} (0) = 0$$

$$\therefore \quad f\left(\frac{1}{2}+0\right) \neq f\left(\frac{1}{2}-0\right) \neq f\left(\frac{1}{2}\right).$$

Hence $f(x)$ is discontinuous at $x = \frac{1}{2}$.

The purpose of taking these two examples was to emphasise the fact that if a function possesses a finite derivative at a point of the interval, then $f(x)$ is continuous at that point but if $f(x)$ possesses an infinite derivative at a point then it may or may not be continuous at that point.

This fact is very important and should be always kept in mind.

Worked Out Examples

Ex.1. Prove that the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

[P.U. 98H, 2002H; M.U. 66H, 68H, 71H, 72H; Bhag. 66H, 67H; R.U. 71H, 90, 91H, 96H; B.U. 94, 98H; Dumka 2000H]

Soln. For continuity of the function at $x = 0$, we have $f(0) = 0$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} |0+h| = 0$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} |0-h| = 0$$

$$\therefore \quad f(0+0) = f(0-0) = f(0).$$

Therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1;$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - 0}{-h}$$

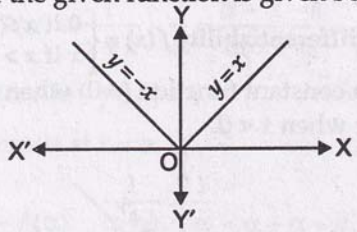
$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

$\therefore \quad Rf'(0) \neq Lf'(0)$.

Hence $f(x)$ is not differentiable at $x = 0$.

By the method of graph :

The graph of the given function is given below :

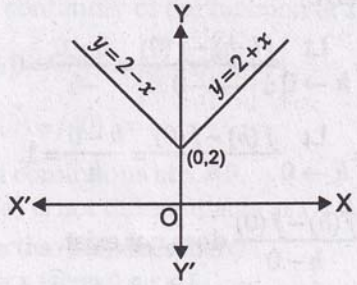


From the figure, it is obvious that the function is continuous at $x = 0$, but it is not differentiable at $x = 0$.

Ex.2. Discuss the continuity and differentiability of the following function at $x = 0$:

$$f(x) = 2 + x \text{ if } x \geq 0$$
$$= 2 - x \text{ if } x < 0.$$ **[B.U. 62H; R.U. 89H]**

Soln. For continuity, we have $f(0) = 2$



$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (2+h) = 2;$$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (2-h) = 2.$$

Since $f(0+0)=f(0-0)=f(0)$, therefore $f(x)$ is continuous at $x=0$. As regards differentiability, we have

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)-2}{h}=1;$$

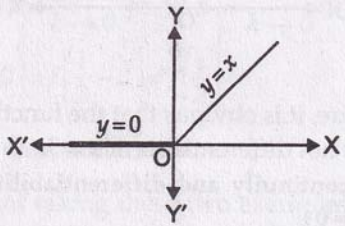
$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{(2+h)-2}{-h}=-1.$$

Since $Rf'(0)\neq Lf'(0)$, therefore the function is not differentiable at $x=0$.

Ex.3. Examine for differentiability $f(x)=\begin{cases}0 \text{ if } x\leq 0\\ x \text{ if } x>0.\end{cases}$ **[M.U. 73H]**

Soln. Since $f(x)$ is a constant function $(=0)$ when $x\leq 0$, therefore it is differentiable when $x<0$.



Also, $f(x)=x$ if $x>0$ and we know that x is a differentiable function. Therefore $f(x)$ is differentiable when $x>0$.

At $x=0$, we have

$$Lf'(0)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h-0}=\frac{0-0}{-h}=0$$

and $$Rf'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\frac{h-0}{h}=1.$$

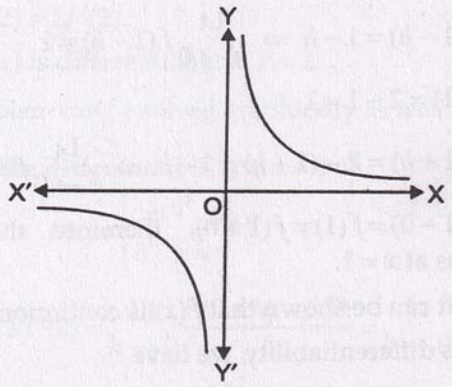
Hence $\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}$ does not exist.

Ex.4. Discuss the continuity and differentiability of the function :

$$f(x)=\frac{1}{x};\ x\neq 0$$

$$=0;\ x=0$$ **[M.U. 68H]**

Soln. Let α be any real number $\neq 0$.



Then $f(\alpha+h)-f(\alpha)=\dfrac{1}{\alpha+h}-\dfrac{1}{\alpha}=\dfrac{\alpha-\alpha-h}{\alpha(\alpha+h)}=\dfrac{-h}{\alpha(\alpha+h)}$

so that $\underset{h\to 0}{\text{Lt}}\{f(\alpha+h)-f(\alpha)\}=0.$

$\therefore$ $f(x)$ is continuous at $x=\alpha$.

Again, $\dfrac{f(\alpha+h)-f(\alpha)}{h}=\dfrac{\dfrac{1}{\alpha+h}-\dfrac{1}{\alpha}}{h}=\dfrac{\alpha-\alpha-h}{h\alpha(\alpha+h)}$

$$=-\frac{h}{h\alpha(\alpha+h)}=-\frac{1}{\alpha(\alpha+h)}$$

so that $\underset{h\to 0}{\text{Lt}}\dfrac{f(\alpha+h)-f(\alpha)}{h}=-\dfrac{1}{\alpha^2}.$

Hence $f(x)$ is differentiable at $x=\alpha$.

We now test the continuity of the function at $x=0$.

For this, $|f(0+h)-f(0)|=\left|\dfrac{1}{h}-0\right|=\left|\dfrac{1}{h}\right|$

so that $\underset{h\to 0}{\text{Lt}}\,|f(h)-f(0)|=\infty.$

Hence $f(x)$ is not continuous at $x=0$.

Consequently $f(x)$ is not differentiable at $x=0$.

Ex.5. Consider the function $f(x)$ defined by

$$f(x)=x \text{ when } 0<x<1$$
$$=2-x \text{ when } 1\le x\le 2$$
$$=x-\frac{1}{2}x^2 \text{ when } x>2.$$

Examine its continuity and differentiability at $x=1$ and $x=2$.

[R.U. 69H]

Soln. For continuity at $x = 1$, we have

$$f(1-h) = 1-h \Rightarrow \lim_{h \to 0} f(1-h) = 1$$

$$f(1) = 2-1 = 1$$

$$f(1+h) = 2-(1+h) = 1-h \Rightarrow \lim_{h \to 0} f(1+h) = 1.$$

Since $f(1-0) = f(1) = f(1+0)$, therefore the function is continuous at $x = 1$.

Similarly it can be shown that $f(x)$ is continuous at $x = 2$.

As regards differentiability, we have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{\{2-(1+h)\}-(2-1)}{h}$$

$$= \lim_{h \to 0} \frac{1-h-1}{h} = \lim_{h \to 0} \frac{-h}{h} = -1;$$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h \to 0} \frac{1-h-1}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = 1.$$

$\therefore \quad Rf'(1) \neq Lf'(1)$.

Hence $f(x)$ is not differentiable at $x = 1$.

Again $Rf'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$

$$= \lim_{h \to 0} \frac{\left\{(2+h)-\frac{1}{2}(2+\quad)\right\}-(2-2)}{h}$$

$$= \lim_{h \to 0} \frac{2+h-\frac{1}{2}(4+4h+h^2)}{h}$$

$$= \lim_{h \to 0} \left\{-1-\frac{h}{2}\right\} = -1.$$

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h \to 0} \frac{2-(2-h)-(2-2)}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1.$$

$\therefore \quad Rf'(2) = Lf'(2).$

Hence $f(x)$ is differentiable at $x = 2$.

Note The problem can be solved graphically as well.

Ex.6. Examine the differentiability at $x = 0$ of

$$f(x) = \begin{cases} e^{-1/x^2}; x \neq 0 \\ 0\ ;\ x = 0 \end{cases}$$ **[R.U. 73H]**

Soln. We have, $Rf'(0) = \underset{h \to 0}{\text{Lt}} \dfrac{f(0+h) - f(0)}{h} = \underset{h \to 0}{\text{Lt}} \dfrac{f(h) - f(0)}{h}$

$$= \underset{h \to 0}{\text{Lt}} \frac{e^{-1/h^2}}{h} = \underset{h \to 0}{\text{Lt}} \frac{1}{he^{1/h^2}}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h} \left[\frac{1}{1 + \frac{1}{h^2} + \frac{1}{2!} \cdot \frac{1}{h^4} + \ldots} \right]$$

$$< \underset{h \to 0}{\text{Lt}} \frac{1}{h} \cdot \frac{1}{1 + \frac{1}{h^2}} = \underset{h \to 0}{\text{Lt}} \frac{h}{h^2 + 1} = 0.$$

Similarly $Lf'(0) = 0$.

Since $Rf'(0) = Lf'(0)$, therefore the function is differentiable at $x = 0$.

Ex.7. Examine the continuity and differentiability of the function

$$f(x) = x \sin \frac{1}{x}; x \neq 0$$

$$f(0) = 0$$

at the point $x = 0$.

[P.U. 50H, 54H, 60H, 66H, 68H, 2002H; B.U. 61H, 67H, 68H, 82H, 91, 93; Bhag.U. 65H, 2001H; R.U. 64H, 74H, 91, 93, 95; M.U. 65H, 66H, 70H, 75H, 91, 93, 2001H; Dumka 98H, 2001H]

Soln. We first of all examine the continuity of the function at $x = 0$.

We have, $f(0) = 0$

$$f(0+0) = \underset{h \to 0}{\text{Lt}} f(0+h)$$

$$= \underset{h \to 0}{\text{Lt}} f(h) = \underset{h \to 0}{\text{Lt}} h \sin \frac{1}{h}$$

$$= 0, \text{ since } \left| \sin \frac{1}{h} \right| \leq 1;$$

and $f(0-0) = \underset{h\to 0}{\text{Lt}} f(0-h) = \underset{h\to 0}{\text{Lt}} f(-h)$

$= \underset{h\to 0}{\text{Lt}} h \sin\frac{1}{h} = 0.$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$. As regards differentiability, we have

$$Rf'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0+h)-f(0)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{h\sin\frac{1}{h}}{h} = \underset{h\to 0}{\text{Lt}} \sin\frac{1}{h}$$ which does not exist

Similarly $Lf'(0)$ does not exist.

Hence $f(x)$ is not differentiable at $x = 0$.

Ex.8. If $f(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$

$f(0) = 0$;

Show that $f(x)$ is continuous and differentiable at $x = 0$.

[P.U. 2003H; M.U. 66H, 76H, 80H, 90, 2001H; B.U. 76H, 92; Bhag.U. 90, 94H, 2002H]

Soln. For continuity of the function at $x = 0$, we have

$$f(0) = 0$$

$$f(0+0) = \underset{h\to 0}{\text{Lt}} f(0+h)$$

$$= \underset{h\to 0}{\text{Lt}} h^2 \sin\frac{1}{h} = 0; \text{ since } \left|\sin\frac{1}{h}\right| \leq 1$$

and $f(0-0) = \underset{h\to 0}{\text{Lt}} f(0-h) = \underset{h\to 0}{\text{Lt}} h^2 \sin\frac{1}{-h} = 0.$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$. As regards differentiability, we have

$$Rf'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{\text{Lt}} \frac{h^2\sin\frac{1}{h}-0}{h}$$

$$= \underset{h\to 0}{\text{Lt}} h\sin\frac{1}{h} = 0;$$

and $Lf'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0-h)-f(0)}{-h} = \underset{h\to 0}{\text{Lt}} \frac{h^2\sin\frac{1}{-h}-0}{-h}$

$$= \underset{h\to 0}{\text{Lt}} h\sin\frac{1}{h} = 0.$$

Hence $Rf'(0) = Lf'(0)$ and consequently $f(x)$ is differentiable at $x = 0$.

Ex.9. Examine the continuity and differentiability of the function $f(x)$ if

$$f(x) = x \cdot \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \text{ when } x \neq 0$$

$$f(0) = 0.$$ **[B.U. 71H; Bhag.U. 72H, 95H; R.U. 90H]**

Soln. Let $x \neq 0$.

If $x \neq 0$, then x, $e^{1/x}$ and $e^{-1/x}$ are continuous and differentiable functions and therefore $f(x)$ is continuous and differentiable when $x \neq 0$.

Now we test the continuity and differentiability of the function at $x = 0$.

We have, $|f(h) - f(0)| = \left| h \dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0 \right|$ ...(1)

As $e^{1/h}$ and $e^{-1/h}$ are both positive, hence

$$|e^{1/h} - e^{-1/h}| < e^{1/h} + e^{-1/h}.$$

Hence from (1), $|f(h) - f(0)| \leq |h|$.

Hence given any $\varepsilon > 0$, there exists a δ satisfying $0 < \delta < \varepsilon$ such that $|f(h) - f(0)| < \varepsilon$ provided $|h - 0| < \delta$.

Hence $f(x)$ is continuous at $x = 0$.

Again, for differentiability at $x = 0$, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0 \right] = \lim_{h \to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1 - 0}{1 + 0} = 1.$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h)}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Thus $Rf'(0) = 1$ and $Lf'(0) = -1$.

Hence $f'(0)$ does not exist.

Hence $f(x)$ is not differentiable at $x = 0$.

Ex.10. Show that the function f defined by

$$f(x) = x^2 \sin \frac{1}{x}; \; x \neq 0$$

$$= 0; \; x = 0.$$

is differentiable for every value of x but the derivative is not continuous at $x = 0$.

[P.U. 99H, 2001H; B.U. 65H, 67H, 76H, 2001H, 2003H; M.U. 64H, 66H, 69H, 73H, 80H, 2002H, 2004H; Bhag. 62H, 63H, 68H; Mithila 96H, 98H; Dumka 94H, 97H, 2002H, 2003H; R.U. 62H, 66H, 68H, 74H]

Soln. Let $x = c \neq 0$. Then in order to determine whether $f(x)$ is differentiable at $x = c$, we shall need to find

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Now $f(c+h) = (c+h)^2 \sin \frac{1}{c+h}$ so that

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^2 \sin \frac{1}{c+h} - c^2 \sin \frac{1}{c}}{h}$$

$$= \frac{(c^2 + 2ch + h^2)}{h} \sin \frac{1}{c+h} - \frac{c^2}{h} \sin \frac{1}{c}$$

$$= \frac{c^2}{h} \left\{ \sin \frac{1}{c+h} - \sin \frac{1}{c} \right\} + (2c+h) \sin \frac{1}{c+h}$$

$$= -\frac{c^2}{h} 2 \cos \frac{2c+h}{2c(c+h)} \cdot \sin \frac{h}{2c(c+h)} + (2c+h) \sin \frac{1}{c+h}$$

$$= -2c^2 \cos \frac{2c+h}{2c(c+h)} \times \frac{\sin \frac{h}{2c(c+h)}}{\frac{h}{2c(c+h)}} \times \frac{1}{2c(c+h)}$$

$$+ (2c+h) \sin \frac{1}{(c+h)}.$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = -2c^2 \times \cos \frac{2c}{2c^2} \times 1 \times \frac{1}{2c^2} + 2c \sin \frac{1}{c}$$

$$= 2c \sin \frac{1}{c} - \cos \frac{1}{c}.$$

$\therefore \quad f'(c) = 2c \sin\frac{1}{c} - \cos\frac{1}{c}$ when $c \neq 0$.

That is, $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ when $x \neq 0$ ...(1)

We know that x and also $\sin\frac{1}{x}$ and $\cos\frac{1}{x}$ are continuous functions of x when $x \neq 0$.

Therefore $f'(x)$ is continuous for $x \neq 0$.

We will show now that $f'(x)$ is not continuous at $x = 0$.

We have $f'(0) = \underset{h \to 0}{\text{Lt}} \frac{f(h) - f(0)}{h}$

$= \underset{h \to 0}{\text{Lt}}\, h \sin\frac{1}{h}$ (as in the previous Example)

$= 0$,

since $\left| h \sin\frac{1}{h} \right| = |h| \left| \sin\frac{1}{h} \right| \leq |h| \to 0$ as $h \to 0$.

Now consider $\underset{x \to 0}{\text{Lt}} f'(x)$ as given by (1).

We know that $\underset{x \to 0}{\text{Lt}}\, x \sin\frac{1}{x} = 0$ and $\cos\frac{1}{x}$ oscillates between -1 and $+1$.

Hence $\underset{x \to 0}{\text{Lt}} f'(x)$ does not exist.

Therefore $f'(x)$ is not continuous at $x = 0$.

Ex.11. Discuss the continuity and differentiability of $f(x)$ at $x = 0$ where

$$f(x) = x^2 \sin\frac{1}{x} + x \cos\frac{1}{x} \text{ when } x \neq 0$$

$$f(x) = 0 \text{ when } x = 0.$$

[M.U. 67H; R.U. 73H; Bhag. 99H]

Soln. For continuity of the function at $x = 0$, we have

$$f(0) = 0$$

$$f(0+0) = \underset{h \to 0}{\text{Lt}} f(0+h) = \underset{h \to 0}{\text{Lt}} f(h)$$

$$= \underset{h \to 0}{\text{Lt}} \left(h^2 \sin\frac{1}{h} + h \cos\frac{1}{h} \right).$$

But $\left| h^2 \sin\frac{1}{h} + h \cos\frac{1}{h} \right| \leq \left| h^2 \sin\frac{1}{h} \right| + \left| h \cos\frac{1}{h} \right|$

$\leq |h^2| + |h|$; $\because \left| \sin\frac{1}{r} \right|, \left| \cos\frac{1}{x} \right| \leq 1$

$$\therefore \quad \lim_{h \to 0} \left| h^2 \sin \frac{1}{h} + h \cos \frac{1}{h} \right| = 0.$$

Thus $f(0+0) = 0$

Again, $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \left\{ -h^2 \sin \frac{1}{h} - h \cos \frac{1}{h} \right\}$$

$$= -\lim_{h \to 0} \left\{ h^2 \sin \frac{1}{h} + h \cos \frac{1}{h} \right\}$$

$= 0$ as above.

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$. As regards differentiability, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} + h \cos \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \left(h \sin \frac{1}{h} + \cos \frac{1}{h} \right).$$

But as we have shown before that $\lim_{h \to 0} \cos \frac{1}{h}$ does not exist.

Hence $Rf'(0)$ does not exist.

Similarly it can be shown that $Lf'(0)$ does not exist

.Hence the function is not differentiable at $x = 0$.

Exercise 2

1. If a function of one variable possesses a finite differential coefficient at a point, show that it is continuous at that point. Is the converse true? Give an example in support of your answer.

   [P.U. 65H; B.U. 61H, 66H; M.U. 66H; R.U. 67H; Bhag.U. 67H; R.U. 85]

2. Show that continuity of a function is a weaker condition than differentiability. **[M.U. 67H, 2001H]**

3. Explain whether $f(x)$ is continuous and has a derivative at the point $x = 1$ where

$$f(x) = x \text{ for } 0 \le x \le 1$$
$$= 2 - x \text{ if } x > 1$$ [Bhag. 2003H; Mithila 97H; Dumka 96H]

4. A function $f(x)$ is defined as follows :

$$f(x) = 1 + 4x \text{ for } x < 1$$
$$= 9 - 4x \text{ for } x \ge 1.$$

Examine the nature of the function at the point $x = 1$ as regards its continuity and differentiability. [B.U. 66H]

5. Discuss the continuity and differentiability of the following function at $x = 0$:

$$f(x) = 5x^2 + 3x + 7; \ -1 < x \le 0$$
$$= 5x^2 - 3x + 7; \ 0 < x < 1.$$ [M.U. 67H]

6. Examine the differentiability of the function

$$f(x) = \frac{1}{x-a} + \frac{1}{x-b} \ (x \ne a, b)$$
$$= 0 \ (x = a, b).$$ [B.U. 68H]

7. Examine the function $f(x) = |x|$ for continuity and differentiability at $x = 0$. [M.U. 66H; Bhag. 66H]

8. (i) Test the derivability of the function $f(x) = x|x|$ at the origin. [Bhag. 92H; 2000H]

(ii) Let $f(x) = \frac{|x|}{x}$, $f(0) = 0$. Discuss the value of $\underset{h \to 0}{\text{Lt}} f(x)$ and differentiability of f at $x = 0$.

9. Show that the function f defined on R by $f(x) = |x| + |x-1|$ is continuous but not differentiable at $x = 0, 1$. [Bhag. 98H, 2004H]

10. Draw the graph of $y = |x-1| + |x-2|$ in the interval $[0, 3]$ and discuss the continuity and differentiability of the function in this interval. [Bhag.U. 90H]

11. Find the right hand derivative and left hand derivative at the origin of the function

$$f(x) = \frac{x}{1 + e^{1/x}} \text{ when } x \ne 0$$
$$= 0 \text{ when } x = 0.$$ [R.U. 94H; Mithila 2000H]

12. Examine the differentiability at $x = 0$ of

$$f(x) = \frac{e^{1/x}}{1 + e^{1/x}} \text{ when } x \neq 0$$

$$= 0 \text{ when } x = 0.$$ [R.U. 95H]

13. Let f be defined on R by setting

$$f(x) = x \cdot \frac{e^{-1/x} - e^{1/x}}{e^{-1/x} + e^{1/x}} \text{ if } x \neq 0$$

$$= 0 \text{ if } x = 0.$$

Show that f is not derivable at $x = 0$.

14. Given that $f(x) = x \cos \frac{1}{x}$ when $x \neq 0$

$$= 0 \text{ when } x = 0.$$

Find $f'(x)$.

15. Discuss the continuity and differentiability of the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} + \cos \frac{1}{x} \text{ when } x \neq 0 \\ 0 \text{ when } x = 0. \end{cases}$$

[P.U. 74H; Bhag. 74H; Dumka 95H, 99H]

16. Examine the continuity and differentiability at $x = 0$ of

$$f(x) = |x| + x \sin \frac{1}{x} \text{ when } x \neq 0$$

$$= 0, \text{ when } x = 0$$ [R.U. 73H]

17. Examine the continuity and differentiability at $x = 0$ of

$$f(x) = |x| + x \cos \frac{1}{x}, \text{ when } x \neq 0$$

$$= 0, \text{ when } x = 0.$$

18. Prove that the function

$$f(x) = x^2 \sin \frac{1}{x} + \alpha x \text{ when } x \neq 0$$

$$= 0 \text{ when } x = 0$$

is continuous and differentiable at $x = 0$. [B.U. 54H]

19. Let a function f be defined on R as follows :

$$f(x) = x^2 \sin \frac{1}{x} \text{ if } x \neq 0$$

$$= 0 \text{ if } x = 0.$$

Show that f is derivable on R but that f' is not continuous at $x = 0$.

20. A function f is defined as follows :

$$f(x) = x^p \cos\frac{1}{x},\ x \neq 0$$

$$f(0) = 0;$$

What conditions should be imposed on p so that

(i) $f(x)$ may be continuous at $x = 0$

(ii) $f(x)$ may have a differential coefficient at $x = 0$. **[P.U. 2005H]**

21. If $f(x) = e^{-1/x^2} \sin\frac{1}{x}$ for $x \neq 0$

$= 0$ for $x = 0$,

show that the function f has at every point a differential coefficient and that it is continuous at $x = 0$. **[P.U. 70H]**

22. If $\phi(x) = \frac{x^2}{a} - a$ for $0 < x < a$

$= a - \frac{a^3}{x^2}$ for $x > a$

$= 0$ for $x = a$;

prove that $\phi(x)$ and $\phi'(x)$ are continuous. **[B.U. 53H]**

23. Let f be defined on R as follows :

$$f(x) = x^3 \sin\frac{1}{x} \text{ if } x \neq 0$$

$$= 0 \text{ if } x = 0.$$

Show that f' is continuous on R but it is not derivable at $x = 0$.

24. Let a function f be defined on R by setting

$$f(x) = x^4 \sin\frac{1}{x} \text{ if } x \neq 0$$

$$= 0 \text{ if } x = 0.$$

Show that $f''(0)$ exists but that f'' is not continuous at $x = 0$.

ANSWERS

3. continuous at $x = 1$ but not differentiable at $x = 1$.

4. continuous at $x = 1$ but not differentiable at $x = 1$.

5. continuous at $x = 0$ but not differentiable at $x = 0$.

6. continuous but not differentiable at $x = a, b$.

7. continuous but not differentiable at $x = 0$.

8. (i) not differentiable at $x = 0$ (ii) not differentiable at $x = 0$

10. continuous at $x = 1, x = 2$ but not differentiable at $x = 1, x = 2$.

11. $f'(0+0)=0, f'(0-0)=1$

12. $f'(0+0)=\infty, f'(0-0)=0$

14. $f(x)$ is not differentiable at $x=0$.

15. neither continuous nor differentiable at $x=0$.

16. continuous at $x=0$ but not differentiable at $x=0$.

17. continuous at $x=0$ but not differentiable at $x=0$.

20. (i) $p>0$ (ii) $p \geq 2$.

□□□

1

SUCCESSIVE DIFFERENTIATION

1.1. We have learnt to determine the first order of derivative of a given function $y = f(x)$ at the Intermediate level. Thus we know how to evaluate $\frac{dy}{dx}$ of algebraic, exponential or inverse functions. In this chapter we shall learn to derive higher orders of derivative of a given function successively. Hence the name successive differentiation. But before doing that, we would like to give a summary of the important formulae and results obtained earlier. They are the following : If

$y = \sin x$; $\frac{dy}{dx} = \cos x$

$y = \cos x$; $\frac{dy}{dx} = -\sin x$

$y = \tan x$; $\frac{dy}{dx} = \sec^2 x$

$y = \cot x$; $\frac{dy}{dx} = -\operatorname{cosec}^2 x$

$y = \sec x$; $\frac{dy}{dx} = \sec x \tan x$

$y = \operatorname{cosec} x$; $\frac{dy}{dx} = -\operatorname{cosec} x . \tan x$

$y = x^n$; $\frac{dy}{dx} = nx^{n-1}$

$y = e^x$; $\frac{dy}{dx} = e^x$

$y = \log x$; $\frac{dy}{dx} = \frac{1}{x}$

$y = \sin^{-1} x$; $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

$y = \cos^{-1} x$; $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$

$y = \tan^{-1} x$; $\frac{dy}{dx} = \frac{1}{1 + x^2}$

$$y = \cot^{-1}x \quad ; \quad \frac{dy}{dx} = -\frac{1}{1+x^2}$$

$$y = \sec^{-1}x \quad ; \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$y = \operatorname{cosec}^{-1}x \quad ; \quad \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

If $y = c$ (constant) ; $\frac{dy}{dx} = 0.$

If $y = uv$, where u, v are functions of x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

So much for the differentiation of well-known functions which we did earlier. However, the differentiation of a function will remain incomplete if we fail to undertake differentiation of hyperbolic functions.

The sine-hyperbolic function and the cosine-hyperbolic function are defined as follows:

$$\sin hx = \frac{e^x - e^{-x}}{2} \; ; \; \cos hx = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are defined as in the case of circular functions viz:

$$\tan hx = \frac{\sin hx}{\cos hx}, \quad \cot hx = \frac{\cos hx}{\sin hx}$$

$$\sec hx = \frac{1}{\cos hx}, \quad \operatorname{cosec} hx = \frac{1}{\sin hx}$$

Here, first we wish to derive the identity connecting $\sin hx$ and $\cos hx$; namely $\cos h^2x - \sin h^2x = 1$.

For this, we find, $\cos h^2x - \sin h^2x$

$$= \left\{\frac{e^x + e^{-x}}{2}\right\}^2 - \left\{\frac{e^x - e^{-x}}{2}\right\}^2$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1.$$

Thus $\cos h^2x - \sin h^2x = 1$.

Further, differentiating the sine-hyperbolic function, we get

$$\frac{d}{dx}\{\sin hx\} = \frac{d}{dx}\left\{\frac{e^x - e^{-x}}{2}\right\} = \frac{e^x + e^{-x}}{2} = \cos hx.$$

Similarly $\frac{d}{dx}\{\cos hx\} = \sin hx$,

$$\frac{d}{dx}\{\tan hx\} = \frac{d}{dx}\left[\frac{\sin hx}{\cos hx}\right]$$

$$= \frac{\cos hx \times (\text{d.c. of } \sin hx) - \sin hx\ (\text{d.c. of } \cos hx)}{\cos h^2 x}$$

by the quotient rule.

$$= \frac{\cos hx \times (\cos hx) - \sin hx \times (-\sin hx)}{\cos h^2 x}$$

$$= \frac{\cos h^2 x - \sin h^2 x}{\cos h^2 x} = \frac{1}{\cos h^2 x} = \sec h^2 x.$$

and so on.

Thus we are fully equipped to cope with the problem which we shall come across in this chapter.

In this chapter, we shall learn to derive the successive differentiation of a function $y = f(x)$.

Let $y = f(x)$ be any function of x. We know that if $y = f(x)$ be differentiated once w.r.t. the independent variable x, then this is denoted by $\frac{dy}{dx}$. The other notation for $\frac{dy}{dx}$ is $f'(x)$. The expression $\frac{dy}{dx}$ is called the first derivative of the function $f(x)$. Similarly if $\frac{dy}{dx}$ be differentiated once again, i.e. $y = f(x)$ be differentiated twice, then that is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. The expression $\frac{d^2y}{dx^2}$ is called the second derivative of the given function $f(x)$.

In the same manner, $\frac{d^3y}{dx^3}$ means that the given function $y = f(x)$ is differentiated thrice. The expression $\frac{d^3y}{dx^3}$ is called the third derivative of the given function; and so on.

The other notation for $\frac{d^3y}{dx^3}$ is $f'''(x)$. Similarly we can determine the fourth derivative; fifth derivative and in a general way the n th derivative by

differentiating successively the given function $y = f(x)$ four times, five times, ...n times. The objective of this chapter is to find out the n th derivative of the given function $y = f(x)$.

1.2. Notation:

It is convenient to use the following notations for the successive derivatives of a given function $y = f(x)$ apart from those mentioned above.

first derivative $\frac{dy}{dx}$ by y_1;

second derivative $\frac{d^2y}{dx^2}$ by y_2;

third defivative $\frac{d^3y}{dx^3}$ by y_3;

.....

nth derivative $\frac{d^ny}{dx^n}$ by y_n..

1.3. nth differential coefficients of some standard functions.

I. nth differential of x^m.

Let $y = x^m$

Then $y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

.....

Let us suppose that

$$y_n = m(m-1)(m-2)(m-3) \ldots\ldots (m-n+1)x^{m-n} \quad \ldots(1)$$

Differentiating again, we get

$$y_{n+1} = m(m-1)(m-2)(m-3) \ldots\ldots (m-n+1)(m-n)x^{m-n-1}$$

which is of the same form as (1) with n replaced by $(n+1)$.

Since (1) is true for $n = 1, 2, 3$, therefore by the method of induction, (1) is true for all n. That is,

if $y = x^m$, then $y_n = m(m-1)(m-2)(m-3) \ldots (m-n+1)x^{m-n}$.

Similarly,

if $y = (ax+b)^m$, then $y_n = m(m-1)(m-2) \ldots (m-n+1)(ax+b)^{m-n}$.

Cor. 1. If $m = n$, then

$$y_n = n(n-1)(n-2)(n-3) \ldots (n-n+1)x^{n-n}$$

$$= n(n-1)(n-2)(n-3) \ldots 3.2.1 = n!.$$

Thus **if $y = x^n$, then $y_n = n!$**.

Hence $y_{n+1} = 0$, $y_{n+2} = 0$, etc.

As a particular case if $y = x^3$, then $y_3 = 3!$; therefore $y_4 = 0$, $y_5 = 0$ etc.

Cor. 2. If $y = (ax + b)^n$, then $y_n = a^n \cdot n!$.

II. nth differential coefficient of $\dfrac{1}{ax+b}$.

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$.

$\therefore \quad y_1 = (-1)(ax+b)^{-2} \times a$

$y_2 = a(-1)(-2)(ax+b)^{-3} \times a$

$= a^2(-1)(-2)(ax+b)^{-3}$,

$y_3 = a^2(-1)(-2)(-3)(ax+b)^{-4} \times a$

$= a^3(-1)(-2)(-3)(ax+b)^{-4}$,

.....

Let us suppose that

$$y_n = a^n(-1)(-2)(-3) \ldots \text{ to } n \text{ factors} \times (ax+b)^{-(n+1)}$$

$$= \frac{a^n(-1)^n\, 1.2.3. \ldots n}{(ax+b)^{n+1}} = \frac{a^n(-1)^n\, n!}{(ax+b)^{n+1}}. \qquad \ldots(2)$$

Differentiating (2) again w.r.t.x, we get

$$y_{n+1} = a^n(-1)^n(n!)(-n-1)(ax+b)^{-n-1-1} \times a$$

$$= a^n(-1)^n(n!)(-1)(n+1) \cdot \frac{a}{(ax+b)^{n+2}}$$

$$= \frac{a^{n+1}(-1)^{n+1}(n+1)!}{(ax+b)^{n+2}}$$

which is of the same form as (2) with n replaced by $n+1$.

Since (2) is true for $n = 1, 2, 3, \ldots$, hence by the method of induction, it is true for all n.

Thus **if $y = \dfrac{1}{ax+b}$, then $y_n = \dfrac{a^n(-1)^n n!}{(ax+b)^{n+1}}$.**

In particlular case if $a = 1$, then $y = \frac{1}{x+b}$;

$$\therefore \quad y_n = \frac{(-1)^n n!}{(x+b)^{n+1}}.$$

III. nth differential coefficient of e^{mx}.

Let $y = e^{mx}$.

$\therefore \quad y_1 = me^{mx}$

$y_2 = m\ me^{mx} = m^2 e^{mx}$

$y_3 = m^2 \cdot me^{mx} = m^3 e^{mx}$.

Let us suppose that $y_n = m^n e^{mx}$ ...(3)

Differentiating again, we get

$y_{n+1} = m^n \cdot me^{mx} = m^{n+1} e^{mx}$,

which is of the same form as (3).

Since (3) is true for $n = 1, 2, 3, \ldots\ldots$, therefore by the method of induction, it is true for all n.

Thus **if $y = e^{mx}$, then $y_n = m^n e^{mx}$.**

Cor. 1. Putting $m = 1$ above, we get $y = e^x$. $\therefore\ y_n = e^x$.

Cor. 2. Let $y = a^{mx}$, then $y = e^{mx \log a}$.

Now if we put $m \log a = k$, then $y = e^{kx}$.

$\therefore$ from the formula, $y_n = k^n e^{kx} = (m \log a)^n e^{(m \log a)x}$

$$= m^n (\log a)^n \cdot a^{mx}.$$

If $m = 1$, then $y = a^x$.

In this case $y_n = (\log a)^n \cdot a^x$.

IV. nth differential coefficient of $\log (ax + b)$.

Let $y = \log (ax + b)$.

Then $y_1 = \frac{1}{ax+b} \times a = a(ax + b)^{-1}$

$y_2 = a \times (-1)\ (ax + b)^{-2} \times a = a^2 \times (-1) \cdot (ax + b)^{-2}$

$y_3 = a^2 \times (-1) \times (-2)(ax + b)^{-3} \times a = a^3(-1)(-2)(ax + b)^{-3}$.

Let us suppose that

$y_n = a^n(-1)(-2)(-3) \ldots$ to $(n - 1)$ factors $\times\ (ax + b)^{-n}$

$$= \frac{a^n(-1)^{n-1} \cdot 1.2.3. \ \ldots \ (n-1)}{(ax+b)^n}$$

$$= \frac{(-1)^{n-1} a^n (n-1)!}{(ax+b)^n}. \qquad \ldots(4)$$

Differentiating (4) again, we get

$$y_{n+1} = (-1)^{n-1} a^n (n-1)! \{(-n)(ax+b)^{-n-1} \times a\}$$

$$= (-1)^n a^{n+1} (n-1)! (n) (ax+b)^{-n-1} = \frac{(-1)^n a^{(n+1)} (n!)}{(ax+b)^{n+1}}$$

which is of the same form as (4), with n replaced by $(n+1)$.

Since (4) is true for $n = 1, 2, 3, \ldots\ldots$, therefore by the method of induction, it is true for all n.

Thus **if $y = \log(ax+b)$, then** $y_n = \dfrac{(-1)^{n-1} a^n (n-1)!}{(ax+b)^n}$.

In particular case, if $a = 1$, then $y = \log(x+b)$.

Hence $y_n = \dfrac{(-1)^{n-1}(n-1)!}{(x+b)^n}$.

V. nth differential coefficient of $\sin(ax+b)$.

(R.U. 64A; P.U. 48S; Mithila 76A)

Let $y = \sin(ax+b)$.

Then $y_1 = a\cos(ax+b) = a\sin\left(ax+b+\dfrac{\pi}{2}\right)$; for $\sin\left(\dfrac{\pi}{2}+\theta\right) = \cos\theta$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax+b+2.\frac{\pi}{2}\right),$$

$$y_3 = a^3\cos\left(ax+b+2.\frac{\pi}{2}\right)$$

$$= a^3\sin\left(ax+b+2.\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^3\sin\left(ax+b+3.\frac{\pi}{2}\right)$$

.....

Let us suppose that $y_n = a^n\sin\left(ax+b+n\dfrac{\pi}{2}\right)$. ...(5)

Differentiating (5) again w.r.t.x, we get

$$y_{n+1} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right) \times a$$

$$= a^{n+1} \sin\left(ax + b + \frac{n\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^{n+1} \sin\left\{ax + b + (n+1)\frac{\pi}{2}\right\}$$

which is of the same form as (5) with n replaced by $(n + 1)$.

Since (5) is true for $n = 1, 2, 3, \ldots\ldots$, therefore by the method of induction, it is true for all n.

Thus **if $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.**

Particular Case 1. If $b = 0$, then $y = \sin ax$.

$$\therefore \quad y_n = a^n \sin\left(ax + \frac{n\pi}{2}\right).$$

2. If $a = 1$, $b = 0$, then $y = \sin x$.

$$\therefore \quad y_n = \sin\left(x + \frac{n\pi}{2}\right).$$

VI. nth differential coefficient of $\cos(ax + b)$. **(B.U. 74A)**

Let $y = \cos(ax + b)$.

Then $y_1 = -a \sin(ax + b) = a \cos\left(ax + b + \frac{\pi}{2}\right)$; for $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$$y_2 = -a^2 \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$= a^2 \cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2 \cos\left(ax + b + 2.\frac{\pi}{2}\right)$$

$$y_3 = -a^3 \sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$= a^3 \cos\left(ax + b + \frac{2\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^3 \cos\left(ax + b + \frac{3\pi}{2}\right)$$

.....

Let us suppose that

$$y_{n+1} = a^n\left\{-\sin\left(ax + b + \frac{n\pi}{2}\right)\right\} \times a$$

$$= a^{n+1}\cos\left(ax + b + \frac{n\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^{n+1}\cos\left\{ax + b + (n+1)\frac{\pi}{2}\right\}$$

which is of the same form as (6), with n replaced by $(n + 1)$.

Since (6) is true for $n = 1, 2, 3, \ldots\ldots$, therefore by the method of induction, it is true for all n.

Thus **if $y = \cos(ax + b)$, then $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$.**

Particular case : 1. If $b = 0$, then $y = \cos ax$.

$$\therefore \quad y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right).$$

2. If $a = 1$, $b = 0$, then $y = \cos x$.

$$\therefore \quad y_n = \cos\left(x + \frac{n\pi}{2}\right).$$

VII. nth differential coefficient of $e^{ax}\sin(bx + c)$.

(P.U. 52S; M.U. 70A, 76S; Bhag. 68A, 71A)

Let $y = e^{ax}\sin(bx + c)$.

Then $y_1 = e^{ax} \times$ d.c. of $\sin(bx + c) + \sin(bx + c) \times$ d.c. of e^{ax}

$= e^{ax} \times b\cos(bx + c) + \sin(bx + c) \times ae^{ax}$

$= e^{ax}\{a\sin(bx + c) + b\cos(bx + c)\}$.

Let $a = r\cos\theta$ and $b = r\sin\theta$

so that $r^2 = a^2 + b^2$ and $\tan\theta = \frac{b}{a}$.

Thus $y_1 = e^{ax}\{r\cos\theta.\sin(bx + c) + r\sin\theta.\cos(bx + c)\}$

$= e^{ax}.r\sin(bx + c + \theta)$.

Similarly differentiating twice, thrice, we get

$y_2 = r \times re^{ax}\sin(bx + c + \theta + \theta)$

$= r^2e^{ax}\sin(bx + c + 2\theta)$

$y_3 = r^3e^{ax}\sin(bx + c + 3\theta)$.

Let us suppose that

$$y_n = r^n e^{ax}\sin(bx + c + n\theta) \qquad \ldots(7)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

Let us suppose that

$$y_n = r^n e^{ax}\sin(bx + c + n\theta) \qquad ...(7)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

Differentiating (7) w.r.t.x, we get

$$\begin{aligned} y_{n+1} &= r^n.e^{ax} \times \{\cos(bx + c + n\theta) \times b\} + \sin(bx + c + n\theta) \times \{ae^{ax}\} \\ &= r^n e^{ax}\{b\cos(bx + c + n\theta) + a\sin(bx + c + n\theta)\} \\ &= r^n e^{ax}\{r\sin\theta\cos(bx + c + n\theta) + r\cos\theta\sin(bx + c + n\theta)\} \\ &= r^n e^{ax}.r\{\sin(bx + c + n\theta + \theta)\} \\ &= r^{n+1}e^{ax}\sin\{bx + c + (n + 1)\theta\} \end{aligned}$$

which is of the same form as (7) with n replaced by $n + 1$.

Since (7) is true for $n = 1, 2, 3, \ldots$ therefore by the method of induction, it is true for all n.

Thus **if $y = e^{ax}\sin(bx + c)$, then $y_n = r^n e^{ax}\sin(bx + c + n\theta)$,**

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

Note : Similarly if $y = e^{ax}\cos(bx + c)$ we shall get exactly in the same manner

$$y_n = r^n e^{ax}\cos(bx + c + n\theta)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

Cor. Let $c = 0$, then (i) $y = e^{ax}\sin bx$.

Hence $y_n = r^n e^{ax}\sin(bx + n\theta)$

$$= (a^2 + b^2)^{\frac{n}{2}}\, e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right);$$

putting the values of r and θ. **(P.U. 50A, 54S; M.U. 69A)**

(ii) $y = e^{ax}\cos bx$.

Hence $y_n = r^n e^{ax}\cos(bx + n\theta)$

$$= (a^2 + b^2)^{\frac{n}{2}}\, e^{ax}\cos\left(bx + n\tan^{-1}\frac{b}{a}\right);$$

putting the values of r and θ. **(P.U. 44S, 56A)**

1.4. Summary

1. If $y = x^m$, then $y_n = m(m-1)(m-2) \ldots (m-n+1)x^{m-n}$
2. If $y = x^n$, then $y_n = n!$.
3. If $y = (ax+b)^m$, then
$$y_n = a^n.m(m-1)(m-2) \ldots (m-n+1)(ax+b)^{m-n}$$
4. If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$
5. If $y = e^{mx}$, then $y_n = m^n e^{mx}$.
6. If $y = a^{mx}$, then $y_n = m^n (\log a)^n a^{mx}$
7. If $y = a^x$, then $y_n = (\log a)^n \cdot a^x$.
8. If $y = \log(ax+b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!.a^n}{(ax+b)^n}$
9. If $y = \sin(ax+b)$, then $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$.
10. If $y = \cos(ax+b)$, then $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
11. If $y = e^{ax}\sin(bx+c)$, then $y_n = r^n e^{ax}\sin(bx+c+n\theta)$ where $r^2 = a^2+b^2$ and $\tan\theta = \frac{b}{a}$.
12. If $y = e^{ax}\cos(bx+c)$, then $y_n = r^n e^{ax}\cos(bx+c+n\theta)$ where $r^2 = a^2+b^2$ and $\tan\theta = \frac{b}{a}$.

Worked Out Examples

EXAMPLE 1. *Find the nth differential coefficient of* $\frac{1}{x+a}$.

SOLUTION : Let $y = \frac{1}{x+a} = (x+a)^{-1}$.

Then $y_1 = (-1)(x+a)^{-2}$

$y_2 = (-1)\{(-2)(x+a)^{-3}\}$

$= (-1)(-2)(x+a)^{-3}$

$y_3 = (-1)(-2)\{(-3)(x+a)^{-4}\}$

$= (-1)(-2)(-3)(x+a)^{-4}$.

Similarly, $y_n = (-1)(-2)(-3) \ldots$ to n factors $\times (x + a)^{-(n+1)}$

$$= \frac{(-1)^n.1.2.3 \ldots n}{(x + a)^{n+1}} = \frac{(-1)^n n!}{(x + a)^{n+1}}.$$

EXAMPLE 2. *Find the nth differential coefficient of* $\log(1 + x)$.

SOLUTION : Let $y = \log(1 + x)$.

Then $y_1 = \dfrac{1}{1 + x} = (1 + x)^{-1}$

$y_2 = (-1)(1 + x)^{-2}$

$y_3 = (-1)(-2)(1 + x)^{-3}$

.....

Similarly $y_n = (-1)(-2)(-3) \ldots$ to $(n - 1)$ factors $\times (1 + x)^{-n}$

$$= \frac{(-1)^{n-1}.1.2.3 \ldots (n-1)}{(1 + x)^n} = \frac{(-1)^{n-1}(n-1)!}{(1 + x)^n}.$$

EXAMPLE 3. *Find the nth differential coefficient of* $\sin 2x \cos 3x$.

SOLUTION : Let $y = \sin 2x \cos 3x$ which is $= \dfrac{1}{2} . 2\sin 2x \cos 3x$

$$= \frac{1}{2}\{\sin (2x + 3x) + \sin(2x - 3x)\} = \frac{1}{2}(\sin 5x - \sin x).$$

$\therefore$ $y_n = \dfrac{1}{2}$ {nth d.c. of $\sin 5x - n$th d.c. $\sin x$}

$$= \frac{1}{2}\left\{5^n \sin\left(5x + \frac{n\pi}{2}\right) - \sin\left(x + \frac{n\pi}{2}\right)\right\}.$$

EXAMPLE 4. *Find the nth differential coefficient of* $e^x \cos x$.

SOLUTION : Let $y = e^x \cos x$.

Then from the formula, $y_n = r^n e^x \cos(x + n\theta)$

where $r^2 = 1^2 + 1^2$ and $\tan\theta = \dfrac{1}{1}$ i.e. $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$.

$\therefore$ $y_n = (\sqrt{2})^n . e^x \cos\left(x + \dfrac{n\pi}{4}\right)$

$$= (2)^{\frac{n}{2}} e^x . \cos\left(x + \frac{n\pi}{4}\right).$$

EXAMPLE 5. *Find the nth differential coefficient of* $e^x \sin^2 x$.

SOLUTION : Here $y = e^x \sin^2 x = e^x \dfrac{1 - \cos 2x}{2}$

$$= \frac{1}{2}e^x - \frac{1}{2}e^x \cos 2x.$$

$\therefore$ From the formula,

$$y_n = \frac{1}{2}e^x - \frac{1}{2}e^x(1^2 + 2^2)^{\frac{n}{2}} \times \cos\left(2x + n\tan^{-1}\frac{2}{1}\right)$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^x(5)^{\frac{n}{2}} \cos(2x + n\tan^{-1}2)$$

$$= \frac{1}{2}e^x\left\{1 - (5)^{\frac{n}{2}} \cos(2x + n\tan^{-1}2)\right\}.$$

EXAMPLE 6. *Find the nth differential coefficient of* $e^{3x} \sin 2x.\cos x$.

SOLUTION : Let $y = e^{3x}\sin 2x\cos x$.

Now $\sin 2x\cos x = \dfrac{1}{2} \times 2\sin 2x\cos x = \dfrac{1}{2}\{\sin(2x + x) + \sin(2x - x)\}$

$$= \frac{1}{2}\{\sin 3x + \sin x\}.$$

$\therefore$ $$y = e^{3x} \cdot \frac{1}{2}\{\sin 3x + \sin x\} = \frac{1}{2}[e^{3x}\sin 3x + e^{3x}\sin x].$$

$\therefore$ $y_n = \dfrac{1}{2}[n\text{th d.c. of } e^{3x}\sin 3x + n\text{th d.c. of } e^{3x}\sin x]$

$$= \frac{1}{2}\left[(3^2 + 3^2)^{\frac{n}{2}} e^{3x}\sin\left(3x + n\tan^{-1}\frac{3}{3}\right)\right]$$

$$+ \left[(3^2 + 1^2)^{\frac{n}{2}} e^{3x}\sin\left(x + n\tan^{-1}\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[(18)^{\frac{n}{2}} e^{3x}.\sin\left(3x + n\frac{\pi}{4}\right) + (10)^{\frac{n}{2}} e^{3x}\sin\left(x + n\tan^{-1}\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}e^{3x}\left[(18)^{\frac{n}{2}} \sin\left(3x + \frac{n\pi}{4}\right) + (10)^{\frac{n}{2}} \sin\left(x + n\tan^{-1}\frac{1}{3}\right)\right].$$

EXAMPLE 7. *If* $y = \sin mx + \cos mx$, *prove that* $y_n = m^n\{1 + (-1)^n\sin 2mx\}^{\frac{1}{2}}$.

(B.U. 62A, 72A, 67; M.U. 64S; P.U. 42S, 63A)

SOLUTION : It is given that $y = \sin mx + \cos mx$.

Therefore from the formula,

$$y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$$

$$= m^n \left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}$$

$$= m^n \left[\left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= m^n \left[\sin^2\left(mx + \frac{n\pi}{2}\right) + \cos^2\left(mx + \frac{n\pi}{2}\right) + 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= m^n \left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}} = m^n [1 + \sin(2mx + n\pi)]^{\frac{1}{2}}$$

$$= m^n [1 + (\sin 2mx \cos n\pi + \cos 2mx \sin n\pi)]^{\frac{1}{2}}$$

$$= m^n [1 + \cos n\pi . \sin 2mx]^{\frac{1}{2}} \quad \ldots(1); \text{ since } \sin n\pi = 0$$

Now $\cos n\pi = -1$, if n is an odd integer

and $\cos n\pi = +1$ if n is an even integer.

Therefore we can write $\cos n\pi = (-1)^n$.

Hence from (1), $y_n = m^n [1 + (-1)^n \sin 2mx]^{\frac{1}{2}}$.

EXAMPLE 8. *If* $ax^2 + 2hxy + by^2 = 1$, *prove that* $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}$

(B.U. 55S, 68H; Bhag. 65S; R.U. 71S)

SOLUTION : Given $ax^2 + 2hxy + by^2 = 1$.

Differentiating w.r.t. x, we get

$$a.2x + 2h\left(x\frac{dy}{ax} + y\right) + b \,.\, 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad ax + \frac{dy}{dx}(hx + by) + hy = 0$$

$$\Rightarrow \quad (ax + hy) + \frac{dy}{dx}(hx + by) = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by}.$$

Again differentiating w.r.t.x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{(hx+by)(a+hy_1)-(ax+hy)(h+by_1)}{(hx+by)^2}\right]; \text{ where } y_1 = \frac{dy}{dx}$$

$$= -\frac{(ahx + h^2xy_1 + aby + bhyy_1) - (ahx + abxy_1 + h^2y + bhyy_1)}{(hx+by)^2}$$

$$= -\frac{(h^2-ab)xy_1 + (ab-h^2)y}{(hx+by)^2}$$

$$= -\frac{(h^2-ab)(xy_1-y)}{(hx+by)^2} = \frac{(h^2-ab)(y-xy_1)}{(hx+by)^2} \qquad ...(1)$$

But $y - xy_1 = y - x \times \left\{-\frac{ax+hy}{hx+by}\right\} = y + \frac{ax^2+hxy}{hx+by}$

$$= \frac{hxy + by^2 + ax^2 + hxy}{hx+by\cdot} = \frac{ax^2+2hxy+by^2}{hx+by}$$

$$= \frac{1}{hx+by}; \quad \text{since } ax^2+2hxy+by^2 = 1.$$

Therefore, from (1)

$$\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^2} \times \frac{1}{hx+by} = \frac{h^2-ab}{(hx+by)^3}$$

EXAMPLE 9. *Prove that* $\dfrac{d^3x}{dy^3} = -\dfrac{\dfrac{d^3y}{dx^3}\dfrac{dy}{dx} - 3\left(\dfrac{d^2y}{dx^2}\right)^2}{\left(\dfrac{dy}{dx}\right)^5}$ **(P.U. 51H)**

SOLUTION : We have $\dfrac{dx}{dy} = \dfrac{1}{\left(\dfrac{dy}{dx}\right)}$.

Differentiating w.r.t. y, we get

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(1/\frac{dy}{dx}\right) = \frac{d}{dx}\left(1/\frac{dy}{dx}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{1}{\left(\dfrac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\dfrac{dy}{dx}} = -\frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}.$$

Again, differentiating w.r.t. y, we get

$$\frac{d^3x}{dy^3} = -\frac{d}{dy}\cdot\left\{\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^3\right\}$$

$$= -\frac{d}{dx}\left\{\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^3\right\}\cdot\frac{dx}{dy}$$

$$= -\frac{\left(\frac{dy}{dx}\right)^3\cdot\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\cdot 3\left(\frac{dy}{dx}\right)^2\cdot\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^6}\cdot\frac{1}{\frac{dy}{dx}}$$

$$= -\frac{\left(\frac{dy}{dx}\right)^2\left\{\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^2\right\}}{\left(\frac{dy}{dx}\right)^6}$$

$$= -\frac{\frac{d^3y}{dx^3}\cdot\frac{dy}{dx} - 3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^5}.$$

Example 1

Find y_n in the following cases :

1. $y = \frac{1}{1+x}$

2. $y = \frac{1}{a-x}$ **(B.U. 7**

3. $y = \frac{1}{(1+x)^2}$

4. $y = \frac{1}{(1-x)^2}$

5. $y = \frac{1}{(ax+b)^m}$

6. $y = \frac{1}{\sqrt{1+x}}$

7. $y = \sin^2 x$

8. $y = \cos^2 x$

9. $y = \sin^3 x$

10. $y = \cos^3 x$

11. $y = \sin x \cos x$

12. $y = \sin x \sin 2x$

13. $y = \cos x \cos 2x$

14. $y = \sin x \cos 2x$

15. $y = \cos x \sin 2x$

16. $y = \sin 2x \cos 3x$

17. $y = \sin x \cos^2 x$

18. $y = \cos x \sin^2 x$

19. $y = \sin^2 x \cos^2 x$

20. $y = \sin^2 x \cos^3 x$

21. $y = e^x \cos 2x$ **22.** $y = e^x \sin 2x$

23. $y = e^x \cos^2 x$ **24.** $y = e^x \sin^3 x$

25. $y = e^{2x} \sin x . \cos x$ **26.** $y = e^x \sin x \sin 2x$

27. If $y = A \sin mx + B \cos mx$, prove that $y_2 = -m^2 y$. **(B.U. 66A)**

28. If $y = A\, e^{mx} + B\, e^{-mx}$, prove that $y_2 = m^2 y$. **(B.U. 66A)**

29. If $y = \log(\sin x)$, prove that $y_3 = \dfrac{2\cos x}{\sin^3 x}$. **(Mithila. 77A)**

30. If $\sqrt{x+y} + \sqrt{y-x} = c$, prove that $y_2 = \dfrac{2}{c^2}$. **(P.U. 61A)**

[**Hint:** Squaring we get

$$x + y + y - x + 2\sqrt{y^2 - x^2} = c^2$$

$$\Rightarrow \quad 2\sqrt{y^2 - x^2} = c^2 - 2y.$$

Again, squaring both sides, we get

$$4(y^2 - x^2) = c^4 - 4c^2 y + 4y^2$$

$$\Rightarrow \quad 4x^2 - 4c^2 y + c^4 = 0 \Rightarrow 8x - 4c^2 y_1 = 0 \Rightarrow c^2 y_1 = 2x.$$

$\therefore \quad c^2 y_2 = 2.$]

31. Prove that $\dfrac{d^3y}{dx^3} = -\dfrac{\dfrac{d^3x}{dy^3} \cdot \dfrac{dx}{dy} - 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5}$.

32. Show that the $(2n)$th differential coefficient of $\cos^{2m} x$ is

$$(-1)^n 2^{-2m+1} \{(2m)^{2n} \cdot C_0 \cos 2mx + (2m-2)^{2n} \cdot C_1 \cos(2m-2)x + \ldots + 2^{2n} \cdot C_{m-1} \cos 2x\}$$

where C_r is the coeff. of x^r in the expansion of $(1+x)^{2m}$ in powers of x. **(P.U. 46H; M.U. 64H)**

[**Hints:** We shall express $\cos^{2m} x$ in series of cosines of multiples of x.

Let $z = \cos x + i \sin x$. Then $\dfrac{1}{z} = \cos x - i \sin x$.

$\therefore \quad z + \dfrac{1}{z} = 2\cos x.$

Hence $(2\cos x)^{2m} = \left(z + \dfrac{1}{z}\right)^{2m}$

$$= z^{2m} + {}^{2m}C_1\, z^{2m-1}\cdot\frac{1}{z} + {}^{2m}C_2 z^{2m-2}\cdot\frac{1}{z^2} + \ldots + \frac{1}{z^{2m}}$$

$$= \left(z^{2m} + \frac{1}{z^{2m}}\right) + {}^{2m}C_1\left(z^{2m-2} + \frac{1}{z^{2m-2}}\right) + \ldots\ldots$$

$$= 2.\cos 2mx + 2.{}^{2m}C_1\cos(2m-2)\,x + \ldots.$$

$$= 2\{\cos 2mx + C_1\cos(2m-2)x + \ldots.\}$$

$$\Rightarrow \cos^{2m}x = 2^{-2m+1}\{\cos 2mx + C_1\cos(2m-2)x + \ldots.\}.$$

The result follows by using $\frac{d^n}{dx^n}(\cos ax) = a^n\cos\left(ax + n\frac{\pi}{2}\right)$.

ANSWERS

1. $\frac{(-1)^n\, n!}{(1+x)^{n+1}}$

2. $\frac{n!}{(1-x)^{n+1}}$

3. $\frac{(-1)^n(n+1)!}{(1+x)^{n+2}}$

4. $\frac{(n+1)!}{(1-x)^{n+2}}$

5. $\frac{(-1)^n m(m+1)(m+2)\ldots(m+n-1)a^n}{(ax+b)^{m+n}}$

6. $\frac{(-1)^n.1.3.5\ldots(2n-1)}{2^n(1+x)^{(2n+1)/2}}$

7. $-2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right)$

8. $2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right)$

9. $\frac{1}{4}\left\{3\sin\left(x + \frac{n\pi}{2}\right) - 3^n\sin\left(3x + \frac{n\pi}{2}\right)\right\}$

10. $\frac{1}{4}\left[3\cos\left(x + \frac{n\pi}{2}\right) + 3^n\cos\left(3x + \frac{n\pi}{2}\right)\right]$

11. $2^{n-1}\sin\left(2x + \frac{n\pi}{2}\right)$

12. $\frac{1}{2}\left\{\cos\left(x + \frac{n\pi}{2}\right) - 3^n\cos\left(3x + \frac{n\pi}{2}\right)\right\}$

13. $\frac{1}{2}\left\{3^n\cos\left(3x + \frac{n\pi}{2}\right) + \cos\left(x + \frac{n\pi}{2}\right)\right\}$

14. $\frac{1}{2}\left\{3^n\sin\left(3x + \frac{n\pi}{2}\right) - \sin\left(x + \frac{n\pi}{2}\right)\right\}$

15. $\frac{1}{2}\left\{3^n\sin\left(3x + \frac{n\pi}{2}\right) + \sin\left(x + \frac{n\pi}{2}\right)\right\}$

16. $\frac{1}{2}\left\{5^n \sin\left(5x+\frac{n\pi}{2}\right)-\sin\left(x+\frac{n\pi}{2}\right)\right\}$

17. $\frac{1}{4}\left\{3^n \sin\left(3x+\frac{n\pi}{2}\right)+\sin\left(x+\frac{n\pi}{2}\right)\right\}$

18. $\frac{1}{4}\left\{\cos\left(x+\frac{n\pi}{2}\right)-3^n \cos\left(3x+\frac{n\pi}{2}\right)\right\}$

19. $-2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right)$

20. $\frac{1}{16}\left\{2\cos\left(x+\frac{n\pi}{2}\right)-3^n \cos\left(3x+\frac{n\pi}{2}\right)-5^n \cos\left(5x+\frac{n\pi}{2}\right)\right\}$

21. $5^{\frac{n}{2}} e^x \cos(2x + n\tan^{-1}2)$

22. $5^{\frac{n}{2}} e^x \sin(2x + n\tan^{-1}2)$

23. $\frac{1}{2}e^x + \frac{1}{2}5^{\frac{n}{2}} e^x \cos(2x + n\tan^{-1}2)$

24. $\frac{3}{4}\cdot 2^{\frac{n}{2}} e^x \sin\left(x+\frac{n\pi}{4}\right)-\frac{1}{4}\cdot 10^{\frac{n}{2}} e^x \cos(3x + n\tan^{-1}3)$

25. $\frac{1}{2}\cdot 8^{\frac{n}{2}} e^{2x}\sin\left(2x+\frac{n\pi}{4}\right)$

26. $\frac{1}{2}\left[2^{\frac{n}{2}} e^x \cos\left(x+\frac{n\pi}{4}\right)-10^{\frac{n}{2}} e^x \cos(3x + n\tan^{-1}3)\right]$

1.5. Formation of Differential Equation

In this section, we shall learn to form an equation by combining the different orders of differential coefficients of a given function.

Here, we shall particularly consider to form an equation by combining the second derivative, first derivative and the given function.

Worked out Examples

EXAMPLE 1. *If* $y = \frac{ax-b}{a-bx}$, *prove that* $2y_1y_3 = 3y_2^2$.

SOLUTION : Here $y = \frac{ax-b}{a-bx}$ $\therefore$ $(a-bx)y = ax - b$.

Now differentiating

$$(a-bx)y_1 + y(-b) = a.$$

Again, differentiating twice,

$$(a - bx)y_3 + y_2(-b) - by_1 = 0$$

$$\Rightarrow \quad (a - bx)y_2 = 2by_1. \qquad \ldots(1)$$

Again, differentiating thrice,

$$(a - bx)y_3 + y_2(-b) = 2by_2$$

$$\Rightarrow \quad (a - bx)y_3 = 3by_2. \qquad \ldots(2)$$

Now dividing (2) by (1),

$$\frac{y_3}{y_2} = \frac{3y_2}{2y_1} \quad \therefore \quad 2y_1y_3 = 3y_2^2.$$

EXAMPLE 2. *If* $y = e^{a\sin^{-1}x}$, *prove that* $(1 - x^2)y_2 - xy_1 = a^2y$.
(R.U. 62A; Bhag. 63S, 67A; P.U. 52S, 53; M.U. 74A, 82)

SOLUTION : Here $y = e^{a\sin^{-1}x}$.

$$\therefore \quad y_1 = e^{a\sin^{-1}x} \times a. \frac{1}{\sqrt{1 - x^2}} \Rightarrow \sqrt{1 - x^2}\,.\,y_1 = ay.$$

Squaring, we get $(1 - x^2)y_1^2 = a^2y^2$.

Again, differentiating twice,

$$(1 - x^2).2y_1 \times y_2 + y_1^2(-2x) = a^2.2yy_1$$

$$\therefore \quad (1 - x^2)y_2 - xy_1 = a^2y.$$

EXAMPLE 3. *If* $y = \sin(m\sin^{-1}x)$, *prove that* $(1 - x^2)y_2 - xy_1 + m^2y = 0$.
(Bhag. 62S; B.U. 53A, 58S, 60A, 68A; P.U. 45S, 50A, 55A, 63A, 65S; (M.U. 75, 76; Kanpur 83, 88)

SOLUTION : Here $y = \sin(m\sin^{-1}x)$.

$$\therefore \quad y_1 = \cos(m\sin^{-1}x) \times m \,.\, \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \sqrt{1 - x^2}\,y_1 = m\cos(m\sin^{-1}x)$$

$$\Rightarrow \quad (1 - x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2\{1 - \sin^2(m\sin^{-1}x)\} = m^2(1 - y^2).$$

Now, differentiating

$$(1 - x^2)2y_1 \times y_2 + y_1^2(-2x) = m^2(-2yy_1)$$

$$\Rightarrow \quad (1 - x^2)y_2 - xy_1 = -m^2y$$

$$\therefore \quad (1 - x^2)y_2 - xy_1 + m^2y = 0.$$

EXAMPLE 4. *If* $y = (\tan^{-1}x)^2$; *prove that* $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$.

(B.U. 61A; P.U. 56S; R.U. 68A, 72A; M.U. 71A)

SOLUTION : Here $y = (\tan^{-1}x)^2$.

$$\therefore \quad y_1 = 2(\tan^{-1}x) \cdot \frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)y_1 = 2\tan^{-1}x$$

$$\Rightarrow \quad (1+x^2)^2 y_1^2 = 4(\tan^{-1}x)^2 = 4y.$$

Again, differentiating twice,

$$(1+x^2)^2.2y_1y_2 + y_1^2 \cdot 2(1+x^2) \times 2x = 4y_1$$

$$\Rightarrow \quad (1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2.$$

Hence the result.

EXAMPLE 5. *If* $x = \cos(\log y)$, *show that* $(1-x^2)y_2 - xy_1 = y$.

(B.U. 53S; M.U. 66A; P.U. 44A, 49A, 59A)

SOLUTION : Here $x = \cos(\log y)$. $\therefore \quad \log y = \cos^{-1}x$.

Therefore differentiating

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \frac{y_1}{y} = -\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1\sqrt{1-x^2} = -y \quad \Rightarrow \quad y_1^2(1-x^2) = y^2.$$

Again, differentiating we get,

$$y_1^2(-2x) + 2y_1y_2(1-x^2) = 2yy_1$$

$$\Rightarrow \quad -xy_1 + (1-x^2)y_2 = y$$

$$\therefore \quad (1-x^2)y_2 - xy_1 = y.$$

EXAMPLE 6. *If* $y = a\cos(\log x) + b\sin(\log x)$, *prove that* $x^2y_2 + xy_1 + y = 0$.

(B.U. 57S, 59S; P.U. 40A, 44S, 54A; R.U. 66S; M.U. 70A)

SOLUTION : Here $y = a\cos(\log x) + b\sin(\log x)$.

$$\therefore \quad y_1 = -a\sin(\log x) \cdot \frac{1}{x} + b\cos(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad xy_1 = -a\sin(\log x) + b\cos(\log x).$$

Again, differentiating,

$$xy_2 + y_1 = -a\cos(\log x) \cdot \frac{1}{x} - b\sin(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x)$$
$$= -[a\cos(\log x) + b\sin(\log x)] = -y.$$
$$\therefore \quad x^2y_2 + xy_1 + y = 0.$$

EXAMPLE 7. *If* $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$, *prove that* $(x^2 - 1)y_2 + xy_1 = n^2y$. **(M.U. 64A; P.U. 51A; R.U. 65S)**

SOLUTION : Here $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$

$$\therefore \quad y_2 = A\{\text{d.c. of } (x + \sqrt{x^2 - 1})^n\} + B \times \{\text{d.c. of } (x - \sqrt{x^2 - 1})^n\}.$$

But d.c. of $(x + \sqrt{x^2 - 1})^n = n(x + \sqrt{x^2 - 1})^{n-1} \times \left\{1 + \frac{1}{2\sqrt{x^2 - 1}} \times 2x\right\}$;

for d.c . of $x = 1$

and d.c. of $\sqrt{x^2 - 1} = \frac{1}{2\sqrt{x^2 - 1}} \times 2x$

$$= n(x + \sqrt{x^2 - 1})^{n-1} \times \left\{1 + \frac{x}{\sqrt{x^2 - 1}}\right\}$$

$$= n(x + \sqrt{x^2 - 1})^{n-1} \times \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}$$

$$= \frac{n(x + \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}.$$

Similarly,

d.c. of $(x - \sqrt{x^2 - 1})^n = n(x - \sqrt{x^2 - 1})^{n-1} \times \left\{1 - \frac{1}{2\sqrt{x^2 - 1}} \times 2x\right\}$

$$= n(x - \sqrt{x^2 - 1})^{n-1} \times \left\{1 - \frac{x}{\sqrt{x^2 - 1}}\right\}$$

$$= -n(x - \sqrt{x^2 - 1})^{n-1} \times \frac{x - \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}$$

$$= -\frac{n(x - \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}.$$

Thus $$y_1 = \frac{An(x + \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}} - \frac{Bn((x - \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}$$

i.e. $\sqrt{x^2 - 1}\,y_1 = An(x + \sqrt{x^2 - 1})^n - Bn(x - \sqrt{x^2 - 1})^n$.

Again differentiating,

$$\frac{1}{2\sqrt{x^2-1}} \times 2xy_1 + \sqrt{x^2-1}\,y_2$$

$$= An \times \text{d.c. of } (x+\sqrt{x^2-1})^n - Bn \times \text{d.c. of } (x-\sqrt{x^2-1})^n$$

$$= An \times \frac{n(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}} + Bn \times \frac{n(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

$$= \frac{An^2(x+\sqrt{x^2-1})^n + Bn^2(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

$$= \frac{n^2\{A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n\}}{\sqrt{x^2-1}}$$

$$= \frac{n^2 y}{\sqrt{x^2-1}}.$$

Now multiplying both sides by $\sqrt{x^2-1}$, we get $xy_1 + (x^2-1)y_2 = n^2y$.

Hence the result.

EXAMPLE 8. *If* $y = (a+bx)e^{-nx}$, *prove that* $\dfrac{d^2y}{dx^2} + 2n\dfrac{dy}{dx} + n^2y = 0$.

SOLUTION : Here $y = (a+bx)e^{-nx}$.

$$\therefore \quad y_1 = (a+bx) \times \text{d.c. of } e^{-nx} + e^{-nx} \times \text{d.c. of } (a+bx)$$

$$= (a+bx)(-n)e^{-nx} + e^{-nx} \cdot b$$

$$= (-n)y + be^{-nx}. \qquad \ldots(1)$$

Again differentiating,

$$y_2 = -ny_1 + b(-n)e^{-nx}.$$

But from (1), $be^{-nx} = y_1 + ny$.

$$\therefore \quad y_2 = -ny_1 + (-n)(y_1 + ny) = -ny_1 - ny_1 - n^2y$$

$$= -2ny_1 - n^2y$$

$$\therefore \quad y_2 + 2ny_1 + n^2y = 0.$$

Hence the result.

EXAMPLE 9. *If* $y = \log\left(\dfrac{x}{a+bx}\right)^x$, *prove that* $x^3y_2 = (y - xy_1)^2$.

(B.U. 64S, 65A, 72A; P.U. 49S, 53A, 64A, 71A, 75A; M.U. 84)

SOLUTION : From the given equation,

$$y = x\log\frac{x}{a+bx} = x\{\log x - \log(a+bx)\}$$

$$\therefore \quad y_1 = \{\log x - \log(a + bx)\} + x\left\{\frac{1}{x} - \frac{b}{a + bx}\right\}$$

$$= \log\frac{x}{a + bx} + x\left\{\frac{a + bx - bx}{x(a + bx)}\right\}$$

$$= \log\frac{x}{a + bx} + \frac{a}{(a + bx)} = \frac{y}{x} + \frac{a}{a + bx}$$

$$\Rightarrow \quad xy_1 - y = \frac{ax}{a + bx} \qquad ...(1)$$

Again, differentiating

$$xy_2 + y_1 - y_1 = \frac{(a + bx) \times (\text{d.c. of } ax) - ax \times \{\text{d.c. of } (a + bx)\}}{(a + bx)^2}$$

$$= \frac{(a + bx)a - ax(b)}{(a + bx)^2}$$

$$\Rightarrow \quad xy_2 = \frac{a^2}{(a + bx)^2}, \qquad ...(2)$$

Now squaring (1),

$$(xy_1 - y)^2 = \frac{x^2a^2}{(a + bx)^2} = x^2.xy_2; \text{ from (2)} \quad \therefore \quad x^3y_2 = (xy_1 - y)^2.$$

More Difficult Problems

EXAMPLE 10. *If* $y = (a + bx)\cos mx + (c + dx)\sin mx$, *prove that*

$$y_4 + 2m^2y_2 + m^4y = 0. \qquad \textbf{(P.U. 60S)}$$

SOLUTION : Differentiating, we get

$$y_1 = (a + bx)(-m\sin mx) + \cos mx \times (b)$$

$$+ (c + dx)(m\cos mx) + \sin mx \times (d)$$

$$= -m(a + bx)\sin mx + m(c + dx)\cos mx + b\cos mx + d\sin mx.$$

Again, differentiating,

$$y_2 = -m\{(a + bx)m\cos mx + \sin mx(b)\}$$

$$+m\{(c + dx)(-m\sin mx) + \cos mx(d)\} + b(-m\sin mx) + dm\cos mx$$

$$= -m^2(a + bx)\cos mx - bm\sin mx$$

$$-m^2(c + dx)\sin mx + dm\cos mx - bm\sin mx + dm\cos mx$$

$$= -m^2\{(a + bx)\cos mx + (c + dx)\sin mx\}$$

$$-2bm\sin mx + 2dm\cos mx$$

$$= -m^2y - 2m(b\sin mx - d\cos mx) \qquad ...(1)$$

Differentiating twice w.r.t. x, we get

$$y_4 = -m^2 y_2 - 2m\{b(-m^2)\sin mx - d(-m^2)\cos mx\}$$

$$= -m^2 y_2 - 2m(-m^2)(b\sin mx - d\cos mx)$$

$$= -m^2 y_2 + 2m^3\left(\frac{-m^2 y - y_2}{2m}\right); \text{ from (1)}$$

$$= -m^2 y_2 + m^2(-m^2 y - y_2)$$

$$= -m^2 y_2 - m^4 y - m^2 y_2$$

$$= -2m^2 y_2 - m^4 y$$

$\therefore \quad y_4 + 2m^2 y_2 + m^4 y = 0.$

EXAMPLE 11. *If* $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, *prove that* $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2b^2}{p^3}$. **(R.U. 71A)**

SOLUTION : Differentiating, we get

$$2p\frac{dp}{d\theta} = a^2(2\cos\theta)(-\sin\theta) + b^2(2\sin\theta)\cos\theta$$

$$= -a^2(2\sin\theta\cos\theta) + b^2(2\sin\theta\cos\theta)$$

$$= (b^2 - a^2)(2\sin\theta\cos\theta) = (b^2 - a^2)\sin 2\theta$$

$$\therefore \quad p\frac{dp}{d\theta} = (b^2 - a^2)\frac{\sin 2\theta}{2} \qquad \ldots(1)$$

Differentiating again, w.r.t. θ, we get

$$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)\times\left(\frac{dp}{d\theta}\right) = \frac{b^2 - a^2}{2}(\cos 2\theta)\times 2$$

$$\Rightarrow \quad p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)\cos 2\theta$$

$$\Rightarrow \quad p^3\frac{d^2p}{d\theta^2} + p^2\left(\frac{dp}{d\theta}\right)^2 = p^2(b^2 - a^2)\cos 2\theta;$$

on multiplying both sides by p^2

$$\Rightarrow \quad p^3\frac{d^2p}{d\theta^2} + \frac{(b^2 - a^2)^2}{4}\sin^2 2\theta = p^2(b^2 - a^2)\cos 2\theta; \text{ from (1)}$$

$$\Rightarrow \quad p^3\frac{d^2p}{d\theta^2} = p^2(b^2 - a^2)\cos 2\theta - \frac{(b^2 - a^2)^2}{4}\sin^2 2\theta$$

$$= p^2(b^2 - a^2)\{\cos^2\theta - \sin^2\theta\} - \frac{(b^2 - a^2)^2}{4}\times 4\sin^2\theta\cos^2\theta.$$

Adding p^4 to both sides,

$$p^4 + p^3\frac{d^2p}{d\theta^2} = p^4 + p^2(b^2 - a^2)(\cos^2\theta - \sin^2\theta) - (b^2 - a^2)^2(\sin^2\theta\cos^2\theta).$$

$$= p^2\{p^2 + (b^2 - a^2)(1 - 2\sin^2\theta)\} - (b^2 - a^2)^2\{\sin^2\theta(1 - \sin^2\theta)\} \quad ...(2)$$

But from the given equation

$$p^2 = a^2(1 - \sin^2\theta) + b^2\sin^2\theta = a^2 + (b^2 - a^2)\sin^2\theta$$

$\therefore \quad (b^2 - a^2)\sin^2\theta = p^2 - a^2.$

Thus (2) becomes

$$p^4 + p^3\frac{d^2p}{d\theta^2} = p^2\{p^2 + (b^2 - a^2) - 2(p^2 - a^2)\} - (b^2 - a^2)^2\{\sin^2\theta - \sin^4\theta)\}$$

$$= p^2\{a^2 + b^2 - p^2\} - (b^2 - a^2)(p^2 - a^2) + (p^2 - a^2)^2$$

$$= (a^2 + b^2)p^2 - p^4 - p^2(b^2 - a^2) + a^2(b^2 - a^2) + p^4 - 2a^2p^2 + a^4$$

$$= p^2\{a^2 + b^2 - b^2 + a^2 - 2a^2\} + a^2b^2 = a^2b^2.$$

$$\therefore \quad p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}.$$

EXAMPLE 12. *If* $y = (a\cos x + b\sin x)e^{-mx}$, *prove that*

$$y_2 + 2my_1 + (m^2 + 1)y = 0. \qquad \textbf{(P.U. 46A)}$$

SOLUTION : Here $y = (a\cos x + b\sin x)e^{-mx}$.

Let $a = k\cos\alpha$, $b = k\sin\alpha$.

$$\therefore \quad k = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\alpha = \frac{b}{a}.$$

Then $y = (k\cos x.\cos\alpha + k\sin x.\sin\alpha)e^{-mx} = k\cos(x - \alpha)e^{-mx}$,

$\therefore \quad y_1 = k[\cos(x - \alpha) \times \text{d.c. of } e^{-mx} + e^{-mx} \times \text{d.c. of } \cos(x - \alpha)$

$$= k[\cos(x - \alpha) \times e^{-mx} \times (-m) + e^{-mx} \times \{-\sin(x - \alpha)\}]$$

$$= k[-me^{-mx}\cos(x - \alpha) - e^{-mx}\sin(x - \alpha)]$$

$$= -k\,e^{-mx}[m\cos(x - \alpha) + \sin(x - \alpha)]. \quad ...(1)$$

Again, differentiating,

$$y_2 = -k[e^{-mx} \times (-m)\{m\cos(x - \alpha) + \sin(x - \alpha)\}$$

$$+ e^{-mx}\{-m\sin(x - \alpha) + \cos(x - \alpha)\}]$$

$$= -k[e^{-mx}\{(-m^2 + 1)\cos(x - \alpha) + (-m - m)\sin(x - \alpha)\}]$$

$$= -\,k\,e^{-mx}[(-m^2 + 1)\cos(x - \alpha) - 2m\sin(x - \alpha)]$$

$$= k\,e^{-mx}[(m^2 - 1)\cos(x - \alpha) + 2m\sin(x - \alpha)] \quad ...(2)$$

First we eliminate $\sin(x-\alpha)$ from (1) and (2). For this, we multiply (1) by $2m$ and then add in (2). Then

$$y_2 + 2my_1 = k\,e^{-mx}[(m^2-1)\cos(x-\alpha) - 2m^2\cos(x-\alpha)]$$

$$= k\,e^{-mx}[(-m^2-1)\cos(x-\alpha)]$$

$$= -k\,e^{-mx}(m^2+1)\cos(x-\alpha) = -(m^2+1)y.$$

$$\therefore \quad y_2 + 2my_1 + (m^2+1)y = 0.$$

Example 2

1. If $y = \dfrac{ax^2+bx+c}{1-x}$, prove that $(1-x)y_3 = 3y_2$. **(P.U. 55S; B.U. 65S)**

2. If $y = \dfrac{ax+b}{a+bx}$, prove that $2y_1y_3 = 3y_2^2$. **(R.U. 65A)**

3. If $y = \sin^{-1}x$, prove that (i) $(1-x^2)y_2 - xy_1 = 0$. **(P.U. 46S)**

   (ii) $\dfrac{y_1}{y_2} = \dfrac{1}{x} - x$ **(M.U. 68)**

4. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 = 2$. **(P.U. 65A, 67A)**

5. If $y = a\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$. **(P.U. 51S)**

6. If $y = a\cos(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$. **(P.U. 54S)**

7. If $y = \sin(\sin x)$, prove that $y_2 + y_1\tan x + y\cos^2 x = 0$. **(P.U. 47A)**

8. If $y = \cos(m\sin^{-1}x)$, prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$.
   (B.U. 56A, M.U. 69A)

9. If $x = \sin(\log y)$, prove that $(1-x^2)y_2 - xy_1 = y$.
   (B.U. 56S; R.U. 66A; Bhag. 70A)

10. If $y = e^{\cos^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 - y = 0$. **(P.U. 66A)**

11. If $y = e^{a\tan^{-1}x}$, prove that $(1+x^2)y_2 + 2xy_1 = ay_1$. **(P.U. 63S; B.U. 69H)**

12. If $y = \{\log(x+\sqrt{x^2+1})\}^2$, prove that $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 2$.

13. If $y = (x+\sqrt{1+x^2})^n$, prove that $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - n^2y = 0$.

14. If $y = (x + \sqrt{x^2 - 1})^m$, prove that $(x^2 - 1)y_2 + xy_1 - m^2y = 0$.

15. If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2 - 1)y_2 + xy_1 - m^2y = 0$.

[Hints : Let $y^{\frac{1}{m}} = u$, then from the given equation

$$u + \frac{1}{u} = 2x. \Rightarrow u^2 - 2ux + 1 = 0.$$

$$\therefore \quad u = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \quad y^{\frac{1}{m}} = x \pm \sqrt{x^2 - 1}$$

$$\therefore \quad y = (x \pm \sqrt{x^2 - 1})^m].$$

16. If $y^{\frac{1}{m}} - y^{-\frac{1}{m}} = 2x$, prove that $(x^2 + 1)y_2 + xy_1 = m^2y$.

17. If $x = \cos h\left(\frac{\log y}{m}\right)$, prove that $(x^2 - 1)y_2 + xy_1 - m^2y = 0$.

(Bhag. 62A; B.U. 54A, 58A, 70A; M.U. 65A, 77A; P.U. 52A, 62A)

[Hints: Differentiating the given equation

$$1 = \sin h\left(\frac{\log y}{m}\right) \cdot \frac{1}{m} \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad my = y_1 \sin h\left(\frac{\log y}{m}\right).$$

Squaring, we get $m^2y^2 = y_1^2 \sin h^2\left(\frac{\log y}{m}\right)$

$$\Rightarrow \quad m^2y^2 = y_1^2\left\{\cos h^2\left(\frac{\log y}{m}\right) - 1\right\}$$

$$\Rightarrow \quad m^2y^2 = y_1^2(x^2 - 1).$$

Differentiating both sides w.r.t. x, we get

$$m^2.2yy_1 = y_1^2(2x) + (x^2 - 1)2y_1y_2$$

$$\Rightarrow \quad m^2y = xy_1 + (x^2 - 1)y_2.]$$

18. If $x = \sin h\left(\frac{\log y}{m}\right)$, prove that $(x^2 + 1)y_2 + xy_1 = m^2y$.

(P.U. 41S; M.U. 71A)

19. If $y = e^{ax}\cos bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$. **(B.U. 61S, 71A)**

20. If $y = Ae^{-ax}\cos(bx + c)$, prove that $y_2 + 2ay_1 + (a^2 + b^2)y = 0$.

(B.U. 57A; M.U. 67A)

1.6. Determination of nth derivative of Algebraic rational functions.

Method : In order to find out the nth derivative of any algebraic rational function, we should first of all find out the partial fractions of that function. Here partial fractions may be of two forms : one form is that in which the denominator contains linear factors only and the other form is that in which the denominator contains the linear as well as quadratic factors. We shall, in the sequel, also indicate the method to determine the partial fractions when the expression is either of the two forms. If the denominator in the partial fraction consists of linear (first degree) expression, then its nth derivative can be found out by direct application of the formula. But if the denominator in the partial fraction consists of an expression of second degree, then its nth derivative is found out by the application of De Moivre's Theorem. The above two methods will be illustrated by the following examples.

Case I. *When the denominator of the given expression contains only real linear factors.*

EXAMPLE 1. *If* $y = \dfrac{x+3}{(x+1)(x+2)}$, *find* y_n.

SOLUTION : First of all, we shall find out the partial fraction of

$$\frac{x+3}{(x+1)(x+2)}.$$

For this, let

$$\frac{x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}.$$

$\therefore \quad A(x+2) + B(x+1) = x+3.$

Putting $x = -1$, $A(1) = -1 + 3 = 2$ $\quad \therefore \ A = 2.$

Again putting $x = -2$, $B(-1) = -2 + 3 = 1$ $\quad \therefore \ B = -1.$

Hence $y = \dfrac{x+3}{(x+1)(x+2)} = \dfrac{2}{x+1} - \dfrac{1}{x+2}.$

$\therefore$ From the formula

$$y_n = \frac{2(-1)^n\, n!}{(x+1)^{n+1}} - \frac{(-1)^n\, n!}{(x+2)^{n+1}}$$

$$= (-1)^n n! \left[\frac{2}{(x+1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right].$$

EXAMPLE 2. *Find y_n when* $y = \dfrac{x^n}{1+x}$. **(P. U. 58A)**

SOLUTION : Let $1 + x = t$ so that $x = t - 1$.

$$\therefore \quad y = \frac{x^n}{1+x} = \frac{(t-1)^n}{t}$$

$$= \frac{t^n - {}^nC_1 t^{n-1} + {}^nC_2 t^{n-2} + \ldots + (-1)^n}{t}$$

$$= t^{n-1} - {}^nC_1\, t^{n-2} + {}^nC_2 t^{n-3} + \ldots + \frac{(-1)^n}{t}$$

$$= (x+1)^{n-1} - {}^nC_1(x+1)^{n-2} + {}^nC_2(x+1)^{n-3} + \ldots + \frac{(-1)^n}{x+1}$$

$$\therefore \quad y_n = 0 - 0 + 0 + \ldots + (-1)^n \frac{(-1)^n\, n!}{(x+1)^{n+1}}$$

$$= \frac{n!}{(x+1)^{n+1}}$$

EXAMPLE 3. *If* $y = \dfrac{x^2}{(x-1)^2(x+2)}$, *find* y_n.

SOLUTION : First of all, we find out the partial fraction of the given expression.

Let $x - 1 = z$, then

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{(1+z)^2}{z^2(3+z)} = \frac{1+2z+z^2}{z^2(3+z)}.$$

Now we divide $1 + 2z + z^2$ by $3 + z$. For this

$$3+z\,)\,1+2z+z^2\,(\,\frac{1}{3}+\frac{5}{9}z$$

$$1+\frac{1}{3}z$$

$$- \quad -$$

$$\frac{5}{3}z+z^2$$

$$\frac{5}{3}z+\frac{5}{9}z^2$$

$$- \quad -$$

$$\frac{4}{9}z^2$$

$$\therefore \quad \frac{1+2z+z^2}{z^2(3+z)} = \frac{1}{z^2}\left[\frac{1}{3}+\frac{5}{9}z+\frac{\frac{4}{9}z^2}{3+z}\right] = \frac{1}{3z^2}+\frac{5}{9z}+\frac{4}{9(3+z)}.$$

$\therefore$ Putting the value of z,

$$y = \frac{1}{3(x-1)^2}+\frac{5}{9(x-1)}+\frac{4}{9(x+2)}.$$

$$\therefore \quad y_n = \frac{(-1)^{n+1}(n+1)!}{3(x-1)^{n+2}}+\frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}}+\frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}.$$

Example 3

Find y_n in the following cases when y is equal to

1. $\dfrac{1}{(x+1)(x+2)}$

2. $\dfrac{5}{(x+2)(x+3)}$

3. $\dfrac{x+1}{(x+2)(x+3)}$

4. $\dfrac{4x+5}{(x+2)(2x+1)}$

5. $\dfrac{1}{6x^2+11x+3}$ **(B.U. 60S, 61S, 64A; M.U. 72A, 75A)**

6. $\dfrac{x^3}{(1-x)^n}$ **(P.U. 57A)**

7. $\dfrac{1}{x^2-a^2}$

8. $\dfrac{x}{x^2-4}$

9. $\dfrac{x}{(x-a)(x-b)}$

10. $\dfrac{x^2}{(x-a)(x-b)}$

11. $\dfrac{x^3}{x^2-1}$ **(P.U. 56S)**

12. $\dfrac{x^4}{(x-1)(x-2)}$ **(P.U. 60A)**

13. $\dfrac{x^3}{(x-1)(x-2)(x-3)}$ **(M.U. 69S)**

14. $\dfrac{x^2}{(x-1)^2(x-2)}$

15. $\dfrac{x^2}{(x+1)^2(x+2)}$

16. $\dfrac{1}{(x-1)^3(x-2)}$

17. $\dfrac{x^2}{(x-1)^3(x-2)}$

Answers

1. $(-1)^n(n)!\left[\dfrac{1}{(x+1)^{n+1}}-\dfrac{1}{(x+2)^{n+1}}\right]$

2. $5(-1)^n(n)!\left[\dfrac{1}{(x+2)^{n+1}}-\dfrac{1}{(x+3)^{n+1}}\right]$

3. $(-1)^n n!\left[\frac{2}{(x+3)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right]$

4. $(-1)^n n!\left[\frac{1}{(x+2)^{n+1}}+\frac{2^{n+1}}{(2x+1)^{n+1}}\right]$

5. $\frac{(-1)^n n!}{7}\left[\frac{3^{n+1}}{(3x+1)^{n+1}}-\frac{2^{n+1}}{(2x+3)^{n+1}}\right]$

6. $\frac{1}{(x-1)^{2n}}\left[\frac{(2n-4)!}{(n-4)!}(x-1)^3+\frac{3(2n-3)!}{(n-3)!}(x-1)^2\right.$
$\left.+\frac{3(2n-2)!}{(n-2)!}(x-1)+\frac{(2n-1)!}{(n-1)!}\right]$

7. $\frac{(-1)^n n!}{2a}\left\{\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right\}$

8. $\frac{1}{2}(-1)^n n!\left[\frac{1}{(x-2)^{n+1}}+\frac{1}{(x+2)^{n+1}}\right]$

9. $\frac{(-1)^n n!}{(a-b}\left\{\frac{a}{(x-a)^{n+1}}-\frac{b}{(x-b)^{n+1}}\right\}$

10. $\frac{(-1)^{n-1} n!}{a-b}\left\{\frac{a^2}{(x-a)^{n+1}}-\frac{b^2}{(x-b)^{n+1}}\right\}$

11. $\frac{1}{2}(-1)^n n!\left\{\frac{1}{(x-1)^{n+1}}+\frac{1}{(x+1)^{n+1}}\right\}$

12. $(-1)^n n!\left\{\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right\}$ when $n>2$.

13. $(-1)^n n!\left[\frac{1}{2}\cdot\frac{1}{(x-1)^{n+1}}-\frac{8}{(x-2)^{n+1}}+\frac{27}{2(x-3)^{n+1}}\right]$

14. $(-1)^n n!\left[\frac{4}{(x-2)^{n+1}}-\frac{3}{(x-1)^{n+1}}-\frac{n+1}{(x-1)^{n+2}}\right]$

15. $(-1)^n n!\left\{\frac{n-1}{(x+1)^{n+2}}-\frac{3}{(x+1)^{n+1}}+\frac{5}{(x+2)^{n+1}}\right\}$

16. $(-1)^{n-1} n!\left[\frac{(n+1)(n+2)}{2(x-1)^{n+3}}+\frac{n+1}{(x-1)^{n+2}}+\frac{1}{(x-1)^{n+1}}-\frac{1}{(x-2)^{n+1}}\right]$

17. $(-1)^{n-1} n!\left[\frac{(n+1)(n+2)}{2(x-1)^{n+3}}+\frac{3(n+1)}{(x-1)^{n+2}}+\frac{4}{(x-1)^{n+1}}-\frac{4}{(x-1)^{n-1}}\right]$

Case II. *When the denominator consists of quadratic factors which can not be resolved further into real linear factors.*

EXAMPLE 4. ***Find y_n if*** $y = \dfrac{1}{x^2 + a^2}$.

(R.U. 62A; Bhag. 63A; B.U. 60A; P.U. 41S, 45S, 49A, 51A, 54S, 55A, 80A; M.U. 68, 72S, 74A; Mithila, 79A)

SOLUTION : Here $y = \dfrac{1}{x^2 + a^2} = \dfrac{1}{(x + ia)(x - ia)}$

$$= \frac{1}{2ia}\left[\frac{1}{x - ia} - \frac{1}{x + ia}\right].$$

$\therefore$ From the formula

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n(n)!}{(x - ia)^{n+1}} - \frac{(-1)^n(n)!}{(x + ia)^{n+1}}\right]$$

$$= \frac{(-1)^n(n)!}{2ia}\left[\frac{1}{(x - ia)^{n+1}} - \frac{1}{(x + ia)^{n+1}}\right].$$

To simplify this, we use De Moivre's Theorem, For this, we put $x = r\cos\theta$ and $a = r\sin\theta$, so that $r^2 = x^2 + a^2$ and $\tan\theta = \dfrac{a}{x}$.

Hence $y_n = \dfrac{(-1)^n n!}{2ia}\left[\dfrac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \dfrac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$

$$= \frac{(-1)^n n!}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{(e^{-i\theta})^{n+1}} - \frac{1}{(e^{i\theta})^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{e^{-i(n+1)\theta}} - \frac{1}{e^{i(n+1)\theta}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}} \times 2i\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{ar^{n+1}} \cdot \sin(n+1)\theta.$$

But $r\sin\theta = a \quad \therefore \quad r = \dfrac{a}{\sin\theta}$.

$$\therefore \quad y_n = \frac{(-1)^n n!}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \cdot \sin(n+1)\theta$$

$$\text{i.e. } y_n = \frac{(-1)^n n!}{a^{n+2}} \cdot \sin^{n+1}\theta \sin(n+1)\theta$$

where $\tan\theta = \frac{a}{x}$ **i.e.** $\theta = \tan^{-1}\frac{a}{x} = \cot^{-1}\frac{x}{a}$.

Cor. If $a = 1$, then $y = \frac{1}{x^2+1}$.

In this case $y_n = (-1)^n n! \sin^{n+1}\theta \sin(n+1)\theta$

where $\theta = \tan^{-1}\frac{1}{x} = \cot^{-1}x$. **(P.U. 64S, 65S)**

At $x = 0$, $\tan\theta = \infty$, i.e. $\theta = \frac{\pi}{2}$.

$$\therefore \quad (y_n)_0 = (-1)^n n! \sin^{n+1}\frac{\pi}{2} \cdot \sin(n+1)\frac{\pi}{2} = (-1)^n n! \sin(n+1)\frac{\pi}{2}.$$

EXAMPLE 5. *If* $y = \tan^{-1}x$; *find* y_n. **(P.U. 45A; B.U. 76A)**

SOLUTION : Here $y = \tan^{-1}x$.

$$\therefore \quad y_1 = \frac{1}{1+x^2} \qquad \ldots(1)$$

Now if we differentiate (1), $(n-1)$ number of times, the direct use of the formula for y_n is made. Therefore y_1 has been first found out after differentiation so that the form of the function on R.H.S. may be amenable to the use of formula. Hence to differentiate (1), $(n-1)$ number of times, we have to put $n-1$ for n in the formula. Hence

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(1)^{n+1}} \sin^n\theta \sin n\theta$$

$$= (-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta \quad \text{where } \theta = \tan^{-1}\frac{1}{x} = \cot^{-1}x.$$

EXAMPLE 6. *If* $y = \tan^{-1}\frac{2x}{1-x^2}$, *find* y_n. **(P.U. 53A; B.U. 67)**

SOLUTION : Since $\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x$, therefore $y = 2\tan^{-1}x$.

$$\therefore \quad y_1 = \frac{2}{1+x^2}$$

$$\therefore \quad y_n = \frac{2(-1)^{n-1}(n-1)!}{(1)^{n+1}} \sin^n\theta \sin n\theta = 2(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$$

where $\theta = \tan^{-1}\frac{1}{x} = \cot^{-1}x$.

EXAMPLE 7. *If* $y = \dfrac{x}{x^2 + a^2}$, *find* y_n. **(P.U. 69S)**

SOLUTION : Here $y = \dfrac{x}{x^2 + a^2} = \dfrac{x}{(x + ia)(x - ia)}$

$$= \frac{1}{2}\left[\frac{1}{x - ia} + \frac{1}{x + ia}\right].$$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x - ia)^{n+1}} + \frac{(-1)^n n!}{(x + ia)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x - ia)^{n+1}} + \frac{1}{(x + ia)^{n+1}}\right].$$

Now as before, put $x = r\cos\theta$ and $a = r\sin\theta$ so that

$r^2 = a^2 + x^2$ and $\tan\theta = \dfrac{a}{x}$.

Hence $y_n = \dfrac{(-1)^n n!}{2}\left[\dfrac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} + \dfrac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} + \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}}\left[\frac{1}{(e^{-i\theta})^{n+1}} + \frac{1}{(e^{i\theta})^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}}\left[e^{i(n+1)\theta} + e^{-i(n+1)\theta}\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}} \cdot 2\cos(n+1)\theta$$

$$= \frac{(-1)^n n!}{r^{n+1}}\cos(n+1)\theta.$$

But $a = r\sin\theta \quad \therefore \quad r = \dfrac{a}{\sin\theta}$.

$$\therefore \quad y_n = \frac{(-1)^n n!}{\left(\dfrac{a}{\sin\theta}\right)^{n+1}}\cos(n+1)\theta$$

$$= \frac{(-1)^n n!}{a^{n+1}} \cdot \sin^{n+1}\theta\cos(n+1)\theta$$

where $\tan\theta = \dfrac{a}{x}$ i.e. $\theta = \tan^{-1}\dfrac{a}{x} = \cot^{-1}\dfrac{x}{a}$.

Cor. If $a = 1$, then $y = \dfrac{x}{x^2+1}$,

In this case, $y_n = (-1)^n n! \sin^{n+1}\theta \cos(n+1)\theta$ where $\tan\theta = \dfrac{1}{x}$.

At $x = 0$, $\tan\theta = \infty$ i.e. $\theta = \dfrac{\pi}{2}$

$\therefore\ (y_n)_0 = (-1)^n n! \cos(n+1)\dfrac{\pi}{2}$.

EXAMPLE 8. *If* $y = \dfrac{1}{(x+1)(x^2+1)}$, *find* y_n.

SOLUTION : First we shall find the P.F.'s of y.

Let $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{(x+1)} + \dfrac{f(x)}{x^2+1}$

$$= \frac{A(x^2+1) + (x+1)f(x)}{(x+1)(x^2+1)}$$

$\Rightarrow\ A(x^2+1) + (x+1)f(x) = 1$

Putting $x + 1 = 0$, i.e., $x = -1$ in (2), we get

$$A(1+1) + 0 = 1 \Rightarrow A = \frac{1}{2}.$$

Now, putting $A = \dfrac{1}{2}$ in (2), we get

$$\frac{1}{2}(x^2+1) + (x+1)f(x) = 1$$

$$\Rightarrow\ (x+1)f(x) = 1 - \frac{1}{2}(x^2+1) = \frac{2 - x^2 - 1}{2}$$

$$= \frac{1-x^2}{2} = \frac{(1-x)(1+x)}{2}$$

$$\therefore\qquad f(x) = \frac{1-x}{2}.$$

Hence, from (1),

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1-x}{x^2+1}$$

Thus, $y = \dfrac{1}{2}\cdot\dfrac{1}{x+1} + \dfrac{1}{2}\cdot\dfrac{1}{x^2+1} - \dfrac{1}{2}\cdot\dfrac{x}{x^2+1}$

Hence, $y_n = \dfrac{1}{2}\cdot\dfrac{(-1)^n n!}{(x+1)^{n+1}} + \dfrac{1}{2}\cdot(-1)^n n! \sin^{n+1}\theta \sin(n+1)\theta$

$$- \frac{1}{2}(-1)^n n! \sin^{n+1}\theta \cos(n+1)\theta \quad \text{where } \tan\theta = \frac{1}{x}.$$

EXAMPLE 9. *If* $y = \dfrac{b + cx}{a + 2bx + cx^2}$, *prove that*

$$y_n = (-1)^n n! \left(\frac{c}{a + 2bx + cx^2}\right)^{\frac{n+1}{2}} \times \left\{\cos(n+1)\tan^{-1}\frac{\sqrt{ac - b^2}}{b + cx}\right\}$$

(M.U. 66H, 68H; P.U. 67H; R.U. 73H)

SOLUTION : We have

$$y = \frac{b + cx}{a + 2bx + cx^2} = \frac{c(b + cx)}{ac + 2bcx + c^2x^2}$$

$$= \frac{c(b + cx)}{(b + cx)^2 + (ac - b^2)}.$$

Let $b + cx = u$ and $ac - b^2 = \alpha^2$.

Then $y = \dfrac{cu}{u^2 + \alpha^2}$.

Therefore from Ex. 7 worked out, we get

$$\frac{d^n y}{du^n} = c\frac{(-1)^n n!}{r^{n+1}}\cos(n+1)\theta,$$

where $u = r\cos\theta$ and $\alpha = r\sin\theta$

so that $r^2 = u^2 + \alpha^2 = c(a + 2bx + cx^2)$.

$\therefore\ r = \{c(a + 2bx + cx^2)\}^{1/2}$

and $\tan\theta = \dfrac{\alpha}{u} = \dfrac{\sqrt{ac - b^2}}{b + cx}$.

Hence $\dfrac{d^n y}{du^n} = c\cdot\dfrac{(-1)^n \cdot n!}{\{c(a + 2bx + cx^2)\}^{(n+1)/2}} \cdot \cos(n+1)\theta$ ...(1)

But $u = b + cx$

$\therefore\ \dfrac{du}{dx} = c$ i.e. $\dfrac{dx}{du} = \dfrac{1}{c}$.

Now $\dfrac{dy}{du} = \dfrac{dy}{dx}\cdot\dfrac{dx}{du} = \dfrac{1}{c}\cdot\dfrac{dy}{dx}$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left\{\frac{1}{c}\cdot\frac{dy}{dx}\right\} = \frac{d}{dx}\left\{\frac{1}{c}\cdot\frac{dy}{dx}\right\}\cdot\frac{dx}{du}$$

$$= \frac{1}{c}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{c} = \frac{1}{c^2}\cdot\frac{d^2y}{dx^2}.$$

Proceeding similarly, we get

$$\frac{d^n y}{du^n} = \frac{1}{c^n}\cdot\frac{d^n y}{dx^n}.$$

Hence, from (1),

$$\frac{1}{c^n}\cdot\frac{d^n y}{dx^n}=\frac{c(-1)^n n!}{\{c(a+2bx+cx^2)\}^{(n+1)/2}}\cos(n+1)\theta$$

$$\therefore\quad \frac{d^n y}{dx^n}=\frac{c^{n+1}(-1)^n n!}{c^{(n+1)/2}(a+2bx+cx^2)^{(n+1)/2}}\times\cos\left\{(n+1)\tan^{-1}\frac{\sqrt{ac-b^2}}{b+cx}\right\}$$

$$=(-1)^n n!\left(\frac{c}{a+2bx+cx^2}\right)^{\frac{n+1}{2}}\times\cos\left\{(n+1)\tan^{-1}\frac{\sqrt{ac-b^2}}{b+cx}\right\}.$$

Example 4

Find y_n in the following cases when y is equal to

1. $\dfrac{1}{x^4+4}$

2. $\dfrac{1}{x^2+16}$

3. $\dfrac{1}{(x^2+a^2)(x^2+b^2)}$

4. $\dfrac{1}{x^2+x+1}$

5. $\dfrac{1}{x^4-a^4}$

6. $\dfrac{1}{(x+1)(x^2+1)}$ **(B.U. 54S**

7. $\tan^{-1}\dfrac{x}{a}$

8. $\cot^{-1}x$

9. $\sin^{-1}\dfrac{2x}{1+x^2}$ **(B.U. 70A)**

10. $\cos^{-1}\dfrac{1-x^2}{1+x^2}$

11. $\tan^{-1}\dfrac{1+x}{1-x}$

12. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$.

(R.U. 59S; R.U. 70A; M.U. 79A; Bhag. 86, 9

[The given exp. $=\dfrac{\pi}{4}+\tan^{-1}x$]

13. $x\tan^{-1}x$.

14. If $x=\cot y$, show that

$$\frac{d^n y}{dx^n}=(-1)^n(n-1)!\sin^n\theta\sin n\theta \text{ where } \theta=\cot^{-1}x.$$

15. If $y=\dfrac{1}{1+x+x^2+x^3}$, show that $(y_n)_0=0$, when n is of the form $4p+2$ or $4p+3$ and is $n!$ or $-n!$ when n is of the form $4p$ or $4p+1$.

ANSWERS

1. $\frac{(-1)^n n!}{2^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \cot^{-1}\frac{x}{2}$

2. $\frac{(-1)^n n!}{4^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \cot^{-1}\frac{x}{4}$

3. $\frac{(-1)^n n!}{a^2 - b^2}\left(\frac{\sin^{n+1}\theta \sin(n+1)\theta}{b^{n+2}} - \frac{\sin^{n+1}\phi \sin(n+1)\phi}{a^{n+2}}\right)$

where $\theta = \cot^{-1}\frac{x}{b}$ and $\phi = \cot^{-1}\frac{x}{a}$

4. $\frac{(-1)^n 2^{n+2} n!}{(\sqrt{3})^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

5. $\frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} - \frac{2}{a^{n+1}}\sin^{n+1}\theta \sin(n+1)\theta\right]$

where $\theta = \cot^{-1}\frac{x}{a}$.

6. $\frac{1}{2}(-1)^n n! \sin^{n+1}\theta\left[\sin(n+1)\theta - \cos(n+1)\theta + \frac{1}{(\cos\theta + \sin\theta)^{n+1}}\right]$

where $\theta = \cot^{-1}x$

7. $\frac{(-1)^{n-1}(n-1)!}{a^n} \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}\frac{x}{a}$

8. $(-1)^{n-2}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

9. $2(-1)^{n-2}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

10. $2(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

11. $(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

12. $\frac{1}{2}(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

13. $(-1)^{n-1}(n-2)! \sin^{n-1}\theta \cos\theta \cos n\theta\{n\tan\theta - \tan n\theta\}$ where $\theta = \cot^{-1}x$

Leibnitz Theorem

Statement : *If u and v are two functions of x which possess derivatives of n^{th} order, then*

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n.$$

(P.U. 76, 81; Mithila 80; R.U. 73, 75, 81, 83; M.U. 77, 78H, 81, 82H, 84, 87; Bhag. 86, 88)

ANSWERS

1. $\frac{(-1)^n n!}{2^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \cot^{-1}\frac{x}{2}$

2. $\frac{(-1)^n n!}{4^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \cot^{-1}\frac{x}{4}$

3. $\frac{(-1)^n n!}{a^2 - b^2}\left(\frac{\sin^{n+1}\theta \sin(n+1)\theta}{b^{n+2}} - \frac{\sin^{n+1}\phi \sin(n+1)\phi}{a^{n+2}}\right)$

where $\theta = \cot^{-1}\frac{x}{b}$ and $\phi = \cot^{-1}\frac{x}{a}$

4. $\frac{(-1)^n 2^{n+2} n!}{(\sqrt{3})^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$ where $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

5. $\frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} - \frac{2}{a^{n+1}} \sin^{n+1}\theta \sin(n+1)\theta\right]$

where $\theta = \cot^{-1}\frac{x}{a}$.

6. $\frac{1}{2}(-1)^n n! \sin^{n+1}\theta\left[\sin(n+1)\theta - \cos(n+1)\theta + \frac{1}{(\cos\theta + \sin\theta)^{n+1}}\right]$

where $\theta = \cot^{-1}x$

7. $\frac{(-1)^{n-1}(n-1)!}{a^n} \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}\frac{x}{a}$

8. $(-1)^{n-2}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

9. $2(-1)^{n-2}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

10. $2(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

11. $(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

12. $\frac{1}{2}(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \cot^{-1}x$

13. $(-1)^{n-1}(n-2)! \sin^{n-1}\theta \cos\theta \cos n\theta\{n\tan\theta - \tan n\theta\}$ where $\theta = \cot^{-1}x$

1.7. Leibnitz Theorem

Statement : *If u and v are two functions of x which possess derivatives of n^{th} order, then*

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n.$$

(P.U. 76, 81; Mithila 80; R.U. 73, 75, 81, 83; M.U. 77, 78H, 81, 82H, 84, 87; Bhag. 86, 88)

From this theorem we know how to determine the nth differential coefficient of the product of two functions. Here we shall give the proof of the theorem in two ways : one by the method of induction and the other by assuming the differential coefficient as an operator.

First proof : *By method of induction*

Let $y = uv$.

where u, v are functions of x.

By directly differentiating successively, we get

$$y_1 = u_1v + uv_1$$

$$y_2 = (u_2v + u_1v_1) + (uv_2 + v_1u_1)$$

$$= u_2v + 2u_1v_1 + uv_2$$

$$= u_2v + {}^2c_1u_1v_1 + {}^2c_2uv_2$$

$$y_3 = (u_3v + u_2v_1) + 2(u_1v_2 + u_2v_1) + (uv_3 + u_1v_2)$$

$$= u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$$

$$= u_3v + {}^3c_1u_2v_1 + {}^3c_2u_1v_2 + {}^3c_3uv_3.$$

Thus we see that this theorem is true for $n = 1, 2, 3$. According to the law of induction, we assume that this theorem is true for $n = m$ and we shall prove that this will also be true for $n = m + 1$ and since this is true for particular values of $n = 1, 2, 3$ ... therefore it will be true for every value of n.

Now we assume that this theorem holds for $n = m$ i.e. we shall get the same formal expression for y_m which will be obtained by putting $n = m$ in the statement of the theorem, that is,

$$y_m = u_mv + {}^mc_1u_{m-1}v_1 + {}^mc_2u_{m-2}v_2 + \cdots$$
$$+ {}^mc_{r-1}u_{m-r+1}v_{r-1} + {}^mc_ru_{n-r}v_r + \cdots$$
$$+ \cdots + {}^mc_{m-1}u_1v_{m-1} + {}^mc_muv_m \qquad \ldots(1)$$

Differentiating once, we get

$$y_{m+1} = (u_{m+1}v + u_mv_1) + {}^mc_1(u_mv_1 + u_{m-1}v_2)$$
$$+ {}^mC_2(u_{m-1}v_2 + u_{m-2}v_3) + \cdots + {}^mC_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}v_r)$$
$$+ {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) +$$
$$\cdots + {}^mC_{m-1}(u_2v_{m-1} + u_1v_m) + {}^mC_m(u_1v_m + uv_{m+1})$$

$$= u_{m+1}v + u_m v_1({}^mC_0 + {}^mC_1) + u_{m-1}v_2({}^mC_1 + {}^mC_2) + \ldots$$
$$+ u_{m-r+1}v_r({}^mC_{r-1} + {}^mC_r) + \ldots$$
$$+ u_1 v_m({}^mC_{m-1} + {}^mC_m) + {}^mC_m uv_{m+1}.$$

But we know that

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r.$$

$\therefore$ putting $r = 1, 2, 3, \ldots$

$${}^mC_0 + {}^mC_1 = {}^{m+1}c_1; \quad {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \text{ etc.}$$

Hence $y_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1}v_2 + \ldots$
$$+ {}^{m+1}C_r u_{m-r+1}v_r + {}^{m+1}C_{m+1}uv_{m+1}.$$

Thus we see that if we assume the theorem to be true for a particular value of $n = m$, then this theorem is also true for the next higher integer $n = m + 1$.

But we have shown before that this theorem is true for $n = 2, 3$, therefore it is true for $n = 4$ and since this is true for $n = 4$, hence this is true for $n = 5$.

Hence this theorem is true for every integral value of n.

Thus the theorem is proved.

Note : If a function is a product of two functions such that one of the factors is of the form x^n, then we shall put $v = x^n$, for as we go on further differentiating x^n; the index of n will succesively decrease by unity, so that proceeding this way the differential of x^n will reduce to zero at one stage.

Second Method :

Let $y = uv$

Then differentiating w.r.t. x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \qquad \ldots(1)$$

Now, we employ three differential operators here.

We denote $\frac{d}{dx}$ by the symbol D.

Then from (1),

$$Dy = uDv + vDu \qquad \ldots(2)$$

Du means that u has been differentiated with respect to x and Dv means that v has been differentiated w.r.t. x. But we assign the following meaning to the operator D:

Du means that only u has been differentiated w.r.t. x while u is independent of v and Dv means that only v has been differentiated w.r.t. x while

v is independent of u. In this situation, we use another symbol $D_1 u$ for Du and another symbol $D_2 v$ for Dv. Thus,

$D_1(uv) = vDu$; for v is independent of u;

$D_2(uv) = uDv$; for u is independent of v.

Hence form (2),

$$Dy = D_1(uv) + D_2(uv) = (D_1 + D_2)uv.$$

$\therefore$ as operators, $D = D_1 + D_2$ ...(3)

Also,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right)y = D(Dy) = D^2y$$

$$\frac{d^3y}{dx^3} = \frac{d^2}{dx^2}\left(\frac{d}{dx}\right).y = \frac{d^2}{dx^2}(Dy) = D^2.Dy = D^3y$$

.....

Generally

$$\frac{d^ny}{dx^n} = D^ny.$$

$\therefore$ $Dy = y_1$

$D^2y = y_2$

$D^3y = y_3$

.....

$D^ny = y_n$.

Now from (1),

$$D^ny = (D_1 + D_2)^n(uv)$$

$$= ({D_1}^n + {}^nC_1 {D_1}^{n-1}D_2 + {}^nC_2 {D_1}^{n-2}{D_2}^2 + \ldots + {}^nC_n {D_2}^n)uv; \text{ from binomial theorem}$$

$$= {D_1}^n(uv) + {}^nC_1 {D_1}^{n-1}D_2(uv) + {}^nC_2 {D_1}^{n-2}{D_2}^2(uv) + \ldots + {}^{n}C_n {D_2}^n(uv)$$

$$= v{D_1}^n(u) + {}^nC_1 {D_1}^{n-1}(u)D_2(v) + {}^nC_2 {D_1}^{n-2}(u)D_2(v) + \ldots + u.\,{}^nC_n {D_2}^n(v)$$

$$= u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots + {}^nC_n uv_n.$$

Thus the theorem is proved.

1.8. Theorem : *To prove that*

$$D^n\{e^{ax}f(x)\} = e^{ax}(D + a)^n f(x) \text{ where } D \equiv \frac{d}{dx}.$$

By using Leibnitz Theorem, we get

$$D^n\{e^{ax}f(x)\} = e^{ax}D^n f(x) + {}^nC_1 . ae^{ax}D^{n-1}f(x) + {}^nC_2 a^2 e^{ax}D^{n-2}f(x) + \ldots + {}^nC_n a^n e^{ax}f(x)$$

$$= e^{ax}\{D^n f(x) + {}^nC_1 aD^{n-1}f(x) + {}^nC_2 a^2 D^{n-2}f(x) + \ldots + {}^nC_n a^n f(x)\}$$

$$= e^{ax}\{D^n + {}^nC_1 aD^{n-1} + {}^nC_2 a^2 D^{n-2} + \ldots + a^n\}f(x)$$

$$= e^{ax}(D + a)^n f(x).$$

In particular; if $f(x) = x^m$, then $D^n\{e^{ax}x^m\} = e^{ax}(D + a)^n x^m$.

Worked out Examples

EXAMPLE 1. *If $y = x^2e^{ax}$, find y_n.* **(P.U. 46A)**

SOLUTION : Here we take $v = x^2$ and $u = e^{ax}$.

$\therefore$ $v_1 = 2x$, $v_2 = 2$, $v_3 = 0$, $v_4 = 0$ ...

$u_1 = ae^{ax}$, $u_2 = a^2e^{ax}$, $u_3 = a^3e^{ax}$ ... $u_n = a^ne^{ax}$

$\therefore$ According to Leibnitz Theorem

$$y_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + {}^nC_3 u_{n-3}v_3 + \ldots$$

$$= a^n e^{ax}x^2 + na^{n-1}e^{ax}(2x) + \frac{n(n-1)}{1\cdot 2}a^{n-2}e^{ax}(2) + 0 + 0 + \ldots$$

$$= a^n e^{ax}x^2 + 2na^{n-1}xe^{ax} + n(n-1)a^{n-2}e^{ax}$$

$$= a^{n-2}e^{ax}\{a^2x^2 + 2nax + n(n-1)\}.$$

EXAMPLE 2. *If $y = x^2\cos x$, find y_n.* **(Bhag. 91)**

SOLUTION : Here we take $v = x^2$ and $u = \cos x$, so that

$$v_1 = 2x, \quad v_2 = 2, \quad v_3 = 0, \quad v_4 = 0\ldots$$

$$u_n = \cos\left(x + \frac{n\pi}{2}\right).$$

$\therefore$ According to Leibnitz Theorem

$$y_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots$$

$$= \cos\left(x + \frac{n\pi}{2}\right)x^2 + {}^nC_1\cos\left\{x + (n-1)\frac{\pi}{2}\right\}\cdot 2x + {}^nC_2\cos\left\{x + \frac{(n-2)\pi}{2}\right\}(2) + 0 + 0 + \ldots$$

$$= x^2\cos\left(x + \frac{n\pi}{2}\right) + 2nx\cos\left\{x + \frac{n-1}{2}\pi\right\} + n(n-1)\cos\left\{x + \frac{(n-2)\pi}{2}\right\}.$$

EXAMPLE 3. *If* $y = x^2 \log x$, *find* y_n; $n \geq 3$. **(B.U. 53S; P.U. 48A; R.U. 68S)**

SOLUTION : Here we take $v = x^2$, $u = \log x$, so that

$$v_1 = 2x, \quad v_2 = 2, \quad v_3 = 0, \quad v_4 = 0 \ldots.$$

$$u_1 = \frac{1}{x}, \quad u_2 = -\frac{1}{x^2}, \quad u_3 = \frac{(-1)(-2)}{x^3}, \ \ldots$$

$$u_n = \frac{(-1)(-2)(-3) \ldots (-\overline{n-1})}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

$\therefore$ According to Leibnitz Theorem

$$y_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot x^2 + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot (2x)$$

$$+ \frac{n(n-1)}{1 \cdot 2} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}(2) + 0 + 0 + \ldots$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + \frac{(-1)^{n-2}(n-2)!}{x^{n-2}} \cdot 2n + \frac{n(n-1)(-1)^{n-3}(n-3)!}{x^{n-2}}$$

$$= \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}[(-1)^2(n-1)(n-2) + (-1)(n-2) \cdot 2n + n(n-1)]$$

$$= \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}[(n-1)(n-2) - 2n(n-2) + n(n-1)]$$

$$= \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}[n^2 - 3n + 2 - 2n^2 + 4n + n^2 - n]$$

$$= \frac{2(-1)^{n-3}(n-3)!}{x^{n-2}} = \frac{2(-1)^{n-1}(n-3)!}{(-1)^2 x^{n-2}}$$

$$= \frac{2(-1)^{n-1}(n-3)!}{x^{n-2}}$$

EXAMPLE 4. *If* $y = e^{a \sin^{-1} x}$, *prove that*

(i) $(1 - x^2)y_2 - xy_1 - a^2 y = 0$,

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0$.

(B.U. 60S, 72A; R.U. 62; M.U. 65A, 70A, 83H, 90, Bhag. 63, 69S, 85)

SOLUTION : (i) Here $y = e^{a \sin^{-1} x}$.

$$\therefore \quad y_1 = e^{a \sin^{-1} x} \times \frac{a}{\sqrt{1 - x^2}} = \frac{ay}{\sqrt{1 - x^2}}$$

$$\Rightarrow \sqrt{1 - x^2}\, y_1 = ay$$

$$\therefore \quad (1 - x^2)y_1^{\,2} = a^2 y^2.$$

Cor. Differentiatng the given function, we get

$y_1 = 2\sin^{-1}x\sqrt{1-x^2}$; let it be u. Then from the result obtained, we find that

$$(1-x^2)u_{n+1} - (2n+1)xu_n - n^2u_{n-1} = 0. \qquad \textbf{(Gorakhpur 81)}$$

EXAMPLE 6. *If* $y = \sin(m\sin^{-1}x)$, *prove that*

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$,

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.

(P.U. 47A, 59A, 61A, 65S, 72A, 86; B.U. 68A, 85, Bhag. 69A; R.U. 63, 67, 70A, 85; M.U. 80, 91, 92H; Gorakhpur 84)

(iii) $\underset{x\to 0}{Lt}\ \dfrac{y_{n+1}}{y_n} = n^2 - m^2$. **(P.U. 61A)**

SOLUTION : We have proved the first part (i) in page 20.

Here we prove the second part.

We have proved that

$$(1-x^2)y_2 - xy_1 + m^2y = 0.$$

Differentiating this according to Leibnitz theorem, we get

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - \{xy_{n+1} + {}^nC_1(1)y_n\} + m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + \{m^2 - n(n-1) - n\}y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0. \qquad \ldots(1)$$

Thus (ii) is proved.

Now taking the limit of (1) as $x \to 0$, we get

$$y_{n+2} = (n^2 - m^2)y_n$$

i.e. $$\underset{x\to 0}{Lt}\ \frac{y_{n+2}}{y_n} = n^2 - m^2.$$

EXAMPLE 7. *If* $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, *prove that*

(i) $(x^2-1)y_2 + xy_1 - m^2y = 0$,

(ii) $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

(P.U. 58H, 69A; R.U. 64H, 82, 84, 87; M.U. 82, 86, 91H; Bhag. 90H; B. U. 94 H; Gorakhpur 85; Kanpur 85)

SOLUTION : We have proved the first part (i) in page 28.

However we give another proof of the same.

Given $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$.

Differentiating it, we get

$$\frac{1}{m}y^{\frac{1}{m}-1}\cdot y_1 + \left(-\frac{1}{m}\right)y^{-\frac{1}{m}-1}\cdot y_1 = 2$$

$$\Rightarrow \quad \frac{1}{m}\cdot\frac{y^{\frac{1}{m}}}{y}\cdot y_1 - \frac{1}{m}\cdot\frac{y^{-\frac{1}{m}}}{y}\cdot y_1 = 2$$

$$\Rightarrow \quad \frac{1}{m}\cdot\frac{y_1}{y}\left[y^{\frac{1}{m}} - y^{\frac{-1}{m}}\right] = 2.$$

Squaring, we get

$$\frac{{y_1}^2}{m^2y^2}\left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right]^2 = 4$$

$$\Rightarrow \quad {y_1}^2\left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4y^{\frac{1}{m}}y^{-\frac{1}{m}}\right] = 4m^2y^2$$

$$\Rightarrow \quad {y_1}^2[4x^2 - 4] = 4m^2y^2$$

$$\Rightarrow \quad (x^2 - 1){y_1}^2 = m^2y^2.$$

Differentiating it again,

$$(x^2 - 1)2y_1y_2 + {y_1}^2(2x) = m^2.2yy_1$$

$$\Rightarrow \quad (x^2 - 1)y_2 + xy_1 = m^2y$$

$$\Rightarrow \quad (x^2 - 1)y_2 + xy_1 - m^2y = 0.$$

Thus (i) is proved.

Again, differentiating n times by using Leibnitz theorem

$$(x^2 - 1)y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + \{xy_{n+1} + {}^nC_1(1)y_n\} - m^2y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + y_n\{n(n-1) + n - m^2\} = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$$

EXAMPLE 8. *Find the nth derivative of x^{2n} in two different ways and show that*

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots = \frac{(2n)!}{(n!)^2}.$$ **(R.U. 70)**

SOLUTION : Let $y = x^{2n}$.

Then $y_n = 2n(2n-1)(2n-2)\ldots(2n-n+1)x^{2n-n}$

$= 2n(2n-1)(2n-2)\ldots(n+1)x^n$

$= \frac{(2n)!}{n!}x^n$ ...(1)

Again, $y = x^n.x^n$.

Therefore applying Leibnitz theorem, we get

$$y_n = n!x^n + {}^nC_1\frac{n!}{1!}x\cdot nx^{n-1} + {}^nC_2\frac{n!}{2!}x^2\cdot n(n-1)x^{n-2} + \ldots$$

$$= n!x^n + n^2.n!x^n + \frac{n(n-1)}{1\cdot 2}\cdot\frac{n!}{1\cdot 2}\cdot n(n-1)x^n + \ldots$$

$$= n!x^n + n!n^2x^n + n!\frac{n^2(n-1)^2}{1^2\cdot 2^2}x^n + \ldots$$

$$= n!x^n\left\{1 + n^2 + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \ldots\right\}$$

...(2)

Hence equating the coeff. of (1) and (2), the result follows.

EXAMPLE 9. *If* $x + y = 1$, *prove that*

$$\frac{d^n}{dx^n}(x^ny^n) = n![y^n - ({}^nC_1)^2y^{n-1}x + ({}^nC_2)^2y^{n-2}x^2 + \ldots + (-1)^nx^n].$$

(P.U. 59H; R.U. 62 H; Bhag. 63H, 92H; M.U. 71H; B.U. 72H; Kanpur 87)

SOLUTION : Since $x + y = 1$, therefore

$$1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1.$$

Now, by Leibnitz theorem, we have

$$\frac{d^n}{dx^n}(x^ny^n) = n!y^n + {}^nC_1\frac{n!}{1!}x.ny^{n-1}(-1) + {}^nC_2\frac{n!}{2!}x^2n(n-1)y^{n-2}(-1)^2$$

$$+ {}^nC_3\frac{n!}{3!}x^3.n(n-1)(n-2)y^{n-3}(-1)^3 + \ldots + {}^nC_n\cdot\frac{n!}{n!}x^n\cdot n!(-1)^n$$

$$= n![y^n - ({}^nC_1)^2xy^{n-1} + ({}^nC_2)^2x^2y^{n-2} + \ldots + (-1)^nx^n]$$

Hence the result.

EXAMPLE 10. *If* $x = \cot y$, *prove that*

$$\frac{d^n}{dx^n}\left(\frac{x^n}{1+x^2}\right) = n!\sin y[\sin y - {}^nC_1\cos y\sin 2y + {}^nC_2\cos^2 y\sin 3y - \ldots]$$

(P.U. 73H; M.U. 75H)

SOLUTION : Let $u = x^n$ and $v = \frac{1}{1+x^2}$.

Then $u_n = n!,\ u_{n-1} = \frac{n!}{1!}x,\ u_{n-2} = \frac{n!}{2!}x^2.$

And $v_n = (-1)^n n! \sin^{n+1}\theta \sin(n+1)\theta,$

where $\tan\theta = \frac{1}{x}$ *i.e.* $\cot\theta = x \ \therefore\ \theta = y.$

Hence $v_n = (-1)^n n! \sin^{n+1} y \sin(n+1)y.$

Putting $n = 1, 2, 3, \ldots$ we get

$$v_1 = -\sin^2 y \sin 2y,\ \ v_2 = 2!\sin^3 y \sin 3y \text{ etc.}$$

Now, applying Leibnitz Theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots$$

$$= n!\sin^2 y + {}^nC_1 \cdot \frac{n!}{1!}x(-\sin^2 y \sin 2y) + {}^nC_2 \cdot \frac{n!}{2!}x^2(2!\sin^3 y \sin 3y) + \ldots$$

$$= n!\sin^2 y - {}^nC_1 \cdot n! \cot y \sin^2 y \sin 2y + {}^nC_2 . n! \cot^2 y \sin^3 y . \sin 3y - \ldots$$

$$= n![\sin^2 y - {}^nC_1 \cos y . \sin y . \sin 2y + {}^nC_2 \cos^2 y \sin y . \sin 3y - \ldots]$$

$$= n! \sin y \{\sin y - {}^nC_1 \cos y \sin 2y + {}^nC_2 \cos^2 y \sin 3y - \ldots\}.$$

EXAMPLE 11. *If* $y = x^n \log x$, *prove that*

$$y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

(P.U. 61A, 68A, 70A, 87; M.U. 69H, 73A, 78A, 86, 93H; R.U. 68S; B.U. 66H, 73A; Mithila 79A; Bhag. 89H; Gorakhpur 83, Kanpur 83)

SOLUTION : Here $y = \frac{d^n}{dx^n}(x^n \log x)$

$$= \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left\{x^n \cdot \frac{1}{x} + n x^{n-1} \log x\right\}$$

$$= \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} + n x^{n-1} \log x)$$

$$= (n-1)! + n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x) \qquad \ldots(1)$$

If we write $I_n\ (= y_n) = \frac{d^n}{dx^n}(x^n \log x)$, then from (1), we have

$$I_n = (n-1)! + n I_{n-1} \qquad \textbf{(P.U. 70A; Kanpur 83)}$$

i.e. $$\frac{I_n}{n!} = \frac{1}{n} + \frac{I_{n-1}}{(n-1)!} \qquad \ldots(2)$$

Now, replacing n successively by $n-1$, $n-2$, $n-3$ ... 2, we get

$$\frac{I_{n-1}}{(n-1)!} = \frac{1}{n-1} + \frac{I_{n-2}}{(n-2)!}$$

$$\frac{I_{n-2}}{(n-2)!} = \frac{1}{n-2} + \frac{I_{n-3}}{(n-3)!}$$

$$\frac{I_{n-3}}{(n-3)!} = \frac{1}{n-3} + \frac{I_{n-4}}{(n-4)!}$$

.....

$$\frac{I_2}{2!} = \frac{1}{2} + \frac{I_1}{1!}.$$

Hence $\frac{I_n}{n!} = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{2} + I_1$

$= \quad " \quad " \quad " \quad " \quad " + \frac{d}{dx}(x\log x)$

$= \quad " \quad " \quad " \quad " \quad " + \left(x\cdot\frac{1}{x} + \log x.1\right)$

$= \quad " \quad " \quad " \quad " \quad " + (1 + \log x)$

$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{2} + 1 + \log x$$

$$\therefore \quad I_n = n!\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1 + \log x\right)$$

i.e. $$y_n = n!\left(\log x + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n}\right).$$

EXAMPLE 12. *Show that*

$$\frac{1}{n!}\frac{d^n}{dx^n}[x^n(\log x)^n] = 1 + S_1\log x + \frac{S_2}{2!}(\log x)^2 + \ldots + \frac{S_n}{n!}(\log x)^n$$

where $S_r \equiv$ the sum of the products taken r at a time of first n natural numbers

(B.U. 59H, 62H; P.U. 60H, 62H, 65H; M.U. 65H, 74H, Bhag. 62H, 66H)

SOLUTION : Let $y = x^n(\log x)^n$.

Put $\log x = z$, so that $x = e^z$ and therefore $y = e^{nz}.z^n$.

Since $x = e^z$ $\quad \therefore \quad \frac{dx}{dz} = e^z$.

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$= \frac{dy}{dz}\cdot e^{-z} = e^{-z}\frac{d}{dz}(y) = e^{-z}D(y) \quad \text{where} \quad D \equiv \frac{d}{dz}.$$

Differentiating again, w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\{e^{-z}D(y)\} = \frac{d}{dz}\{e^{-z}D(y)\}.\frac{dz}{dx}$$

$$= \{e^{-z}.D^2(y) + D(y)(-1)e^{-z}\}.e^{-z}$$

$$= e^{-2z}\{D^2(y) - D(y)\}$$

$$= e^{-2z}(D^2 - D)y = e^{-2z}D(D-1)y.$$

Proceeding in the similar manner, we get

$$\frac{d^ny}{dx^n} = e^{-nz}\{D(D-1)(D-2)\ \ldots\ (D-n+1)\}y.$$

Now $\dfrac{1}{n!}\dfrac{d^n}{dx^n}\{x^n(\log x)^n\}$

$$= \frac{1}{n!}\frac{d^ny}{dx^n}$$

$$= \frac{1}{n!}e^{-nz}\{D(D-1)(D-2)\ \ldots\ (D-n+1)\}e^{nz}z^n$$

$$= \frac{1}{n!}e^{-nz}\,e^{nz}\{(D+n)(D+n-1)\ \ldots\ (D+1)\}z^n;$$

from theorem 1.8

$$= \frac{1}{n!}\{(D+1)(D+2)\ \ldots\ (D+n-1)(D+n)\}z^n$$

$$= \frac{1}{n!}\{D^n + (1+2+3+\ \ldots\ +n)D^{n-1}$$

$$+ (1\cdot2 + 2\cdot3 + \ldots)D^{n-2} + \ \ldots\ + \ \ldots\ (1\cdot2\cdot3\ \ldots\ n)\}z^n$$

$$= \frac{1}{n!}\{D^n + S_1D^{n-1} + S_2D^{n-2} + \ \ldots\ + S_n\}z^n$$

$$= \frac{1}{n!}\left\{n! + S_1\cdot\frac{n!}{1!}z + S_2\cdot\frac{n!}{2!}z^2 + \ \ldots\ + S_n\frac{n!}{n!}z^n\right\}$$

$$= 1 + S_1(\log x) + \frac{S_2}{2!}(\log x)^2 + \ \ldots\ + \frac{S_n}{n!}(\log x)^n.$$

EXAMPLE 13. *If* $y = (x^2-1)^n$, *show that*

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

Hence if $P(x) = \dfrac{1}{2^n\,\lfloor n}\dfrac{d^n}{dx^n}\{(x^2-1)^n\}$, *show that*

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP}{dx}\right\} + n(n+1)P = 0.$$

(M.U. 63H, 78H, 80H; R.U. 66H; P.U. 72H; Bhag, 91H, 94H; Kanpur 88)

SOLUTION : Let $y = (x^2 - 1)^n$. ...(1)

Then $P = \dfrac{d^n y}{dx^n}$.

Differentiating (1), $\dfrac{dy}{dx} = n(x^2 - 1)^{n-1} 2x$.

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2 - 1)\frac{dy}{dx} = 2nx(x^2 - 1)^n = 2nxy \qquad ...(2)$$

Now, differentiating (2), $(n + 1)$ number of times by Leibnitz theorem, we get

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}C_1(2x)\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_2(2)\frac{d^n y}{dx^n}$$

$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_1.(1)\frac{d^n y}{dx^n}\right]$$

$$\Rightarrow \quad (x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + \{2(n+1)x - 2nx\}\frac{d^{n+1}y}{dx^{n+1}}$$

$$+ \{(n+1)n - 2n(n+1)\}\frac{d^n y}{dx^n} = 0$$

$$\Rightarrow \quad (x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^n y}{dx^n} = 0 \qquad ...(3)$$

This proves the first part.

If we put $P = \dfrac{1}{2^n \lfloor n}\dfrac{d^n}{dx^n}(x^2 - 1)^n = \dfrac{1}{2^n \lfloor n}\dfrac{d^n y}{dx^n}$, (3) becomes

$$(x^2 - 1)\frac{d^2P}{dx^2} + 2x\frac{dP}{dx} - n(n+1)P = 0.$$

This $\Rightarrow$ that

$$\frac{d}{dx}\left\{(x^2 - 1)\frac{dP}{dx}\right\} - n(n+1)P = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left\{(1 - x^2)\frac{dP}{dx}\right\} + n(n+1)P = 0.$$

Hence the result.

Cor. If $P = \dfrac{d^n}{dx^n}\{(x^2 - 1)^n\}$, then

$$\frac{d}{dx}\{(1 - x^2)\frac{dP}{dx}\} + n(n+1)P = 0.$$

EXAMPLE 14. *If* $P_n(x) = \dfrac{1}{2^n n!}\cdot\dfrac{d^n}{dx^n}\{(x^2 - 1)^n\}$, *prove the relation*

$$P_{n+1}' = xP_n' + (n+1)P_n \qquad \textbf{(P.U. 70H)}$$

SOLUTION : Given $P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\{(x^2-1)^n\}$

$$\therefore \quad P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} \cdot \frac{d^{n+1}}{dx^{n+1}}\{(x^2-1)^{n+1}\}$$

$$= \frac{1}{2^{n+1}(n+1)!} \cdot \frac{d^n}{dx^n}\left[\frac{d}{dx}\{(x^2-1)^{n+1}\}\right]$$

$$= \frac{1}{2^{n+1}(n+1)!} \cdot \frac{d^n}{dx^n}[(n+1)(x^2-1)^n \cdot 2x]$$

$$= \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\{x(x^2-1)^n\}$$

which by using Leibnitz theorem becomes

$$= \frac{1}{2^n n!}\left[x\frac{d^n}{dx^n}(x^2-1)^n + {}^nC_1(1)\frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\right]$$

$$\Rightarrow \quad P_{n+1}(x) = x \cdot \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\{(x^2-1)^n\} + \frac{n}{2^n n!} \cdot \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}$$

$$= x.P_n + \frac{n}{2^n n!} \cdot \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}.$$

Therefore differentiating both sides w.r.t. x, we get

$$P_{n+1}' = xP_n' + P_n + \frac{n}{2^n n!} \cdot \frac{d^n}{dx^n}\{(x^2-1)^n\}$$

$$= xP_n' + P_n + nP_n$$

$$= xP_n' + (n+1)P_n.$$

Hence the result.

Example 5

Find y_n in the following cases when y is equal to

1. xe^x
2. x^2e^{2x}
3. $x^2\sin x$ **(B.U. 58A)**
4. $x^2\log x^2$
5. $x^2\sin^2 x$
6. $x^2\cos^2 x$
7. (i) If $y = \tan^{-1}x$, then prove that

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0. \quad \textbf{(Bhag. 65A; M.U. 81H)}$$

(ii) If $y = e^{m\cos^{-1}x}$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0. \quad \textbf{(B.U. 92H)}$$

(iii) If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

show that (i) $(1 + x^2)y_1 = my$.

(ii) $(1 + x^2)y_2 + (2x - m)y_1 = 0$ **(Bhag. 92)**

(iii) $(n + 1)a_{n+1} + (n - 1)a_{n-1} = ma_n$.

8. If $y = \sin^{-1}x$, then prove that

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0.$$

(Bhag. 64A, 66A; Mithila. 76A)

9. If $y = (x^2 - 1)^n$, then prove that

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n + 1)y_n = 0.$$

(R.U. 72S; M.U. 78H, 80H; Gorakhpur 82, 86; Kanpur 88)

10. If $y = \log(x + \sqrt{1 + x^2})^2$, prove that

$$(1+ x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0.$$

11. If $y = a\cos(\log x) + b\sin(\log x)$, prove that

$$x^2y_{n+2} + (2n + 1)xy_{n+1} + (n^2 + 1)y_n = 0.$$

(B.U. 63A, 69S, 86; M. U. 66S, 71A, 90H; R.U. 72H; P.U. 70A, 80A, 88)

12. If $y = e^{x^2}$, prove that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$. **(P.U. 55S)**

13. If $y = x\sin x$, prove that $y_n = x\sin\left(x + \frac{n\pi}{2}\right) - n\cos\left(x + \frac{n\pi}{2}\right)$.

14. If $y = x\sin x$, prove that $y_5 - y_1 = 4\frac{y}{x}$.

15. If $y = e^{-x}\cos x$, prove that $y_4 + 4y = 0$.

16. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{a}\right)^n$, prove that

$$x^2y_{n+2} + (2n + 1)xy_{n+1} + 2n^2y_n = 0.$$

(P.U. 66H; R.U. 71H; Bhag. 87; Kanpur 86)

[Hint: We differentiate directly and after simplification, we get

$$x^2y_1^{\,2} = n^2(b^2 - y^2).$$

Differentiating further we get $x^2y_2 + xy_1 + n^2y = 0$.]

17. If $x = \cos h\left(\frac{1}{m}\log y\right)$, prove that

$$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0.$$

18. If $y = x^2e^x$, prove that

$$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

(M.U. 67H; Gorakhpur 81)

19. If $y = x^4 \log x$, prove that $y_6 = -\frac{4!}{x^2}$.

20. If $y = x^4 \log(x^{2/3})$, prove that $\frac{d^6y}{dx^6} + \frac{16}{x^2} = 0$.

21. If $y = x^4 \log x$, prove that $y_n = \frac{(-1)^{n-1}(n-5)!}{x^{n-4}} \cdot 24, \quad n \geq 5.$ **(P.U. 65A)**

22. If $y = x^{n-1} \log x$, prove that $y_n = \frac{(n-1)!}{x}$. **(P.U. 68A; R.U. 70S; M.U. 78)**

23. If $y = x^3 \log x$, find y_n. **(P.U. 58S; R.U. 68S)**

24. If $y = \frac{x^n}{1+x}$, find y_n. **(P.U. 58A)**

[Write $1 + x = t$]

25. If $y = x^2e^{2x}\sin x$, find y_n. **(P.U. 57S)**

[Take $u = e^{2x}\sin x$ and $v = x^2$]

26. If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}\left(\log x - \sum_{r=1}^{n} \frac{1}{r}\right)$. **(P.U. 66S)**

27. If $y = \sin h(m \sin h^{-1}x)$, prove that

$$(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0. \quad \textbf{(B.U. 58H, 63H)}$$

[Use $\frac{d}{dx}\sin h^{-1}x = \frac{1}{\sqrt{1+x^2}}$]

ANSWERS

1. $(x + n)e^x$

2. $e^{2x}2^{n-2}[4x^2 + 4nx + n(n-1)]$

3. $x^2\sin\left(x + \frac{n\pi}{2}\right) + 2nx\sin\left\{x + \frac{(n-1)\pi}{2}\right\} + n(n-1)\sin\left\{x + \frac{(n-2)\pi}{2}\right\}$

4. $\frac{4(-1)^{n-1}(n-3)!}{x^{n-2}}$

5. $-2^{n-3}\left[4x^2\cos\left(2x+\frac{n\pi}{2}\right)+4nx\cos\left\{2x+\frac{(n-1)\pi}{2}\right\}\right.$

$\left.+\,n(n-1)\cos\left\{2x+\frac{(n-2)\pi}{2}\right\}\right]$

6. $2^{n-3}\left[4x^2\cos\left(2x+\frac{n\pi}{2}\right)+4nx\cos\left\{2x+\frac{(n-1)\pi}{2}\right\}\right.$

$\left.+\,n(n-1)\cos\left\{2x+\frac{(n-2)\pi}{2}\right\}\right]$

23. $\frac{(-1)^n 6(n-4)!}{x^{n-3}}$

24. $\frac{n!}{(x+1)^{n+1}}$

25. $5^{\frac{n}{2}}e^{2x}\sin\left\{x+n\tan^{-1}\frac{1}{2}\right\}\cdot x^2+n5^{\frac{n-1}{2}}e^{2x}\sin\left\{x+(n-1)\tan^{-1}\frac{1}{2}\right\}\cdot 2x$

$+\,n(n-1).5^{\frac{n-2}{2}}e^{2x}\sin\left\{x+(n-2)\tan^{-1}\frac{1}{2}\right\}.$

2

EXPANSION

2.1. Let $y = f(x)$ be any function of x.

In this chapter we shall learn how to expand $f(x)$ in asscending powers of x, which literally means approximating a given function by means of a polynomial in x. For this, ordinarily the following four methods are used.

(i) By using Taylor's or Maclaurin's Theorem,

(ii) By using the series obtained from either Algebra or Trigonometry,

(iii) By using Differential Equation,

(iv) By differentiating the known series.

We shall illustrate the above four methods one by one. However, Taylor's series occupies a fundamental position in any scheme of expansion. Hence we begin with Taylor's series. In this chapter, elementary treatment of Taylor's series will be given and its rigorous treatment will be deferred until the next session.

Taylor's and Maclaurin's Series

2.2. Taylor's Theorem (*Elementary treatment*)

The general form of statement of Taylor's series can be given in the following way :

Statement : *Let $f(x)$ be a function of x. If the function $f(x + h)$ can be expanded in a convergent series of positive integral powers of h, then that expansion is*

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots + \frac{h^n}{n!} f^n(x) + \ldots$$

From the expansion of the given function we see that for the truth of the statement of the theorem, the differentials $f'(x), f''(x), f'''(x), \ldots f^n(x)$, should exist finitely.

In fact the more rigorous statement of the theorem reads as follows:

If a function f possesses derivatives of every order in the closed interval $[a, a + h]$ and the Taylor's remainder $R_n \to 0$ as $n \to \infty$, then

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^n(a) + \ldots$$

To illustrate this point, we make the following observation. It will be established by the application of Rolle's Theorem that :

If a function f defined on the closed interval $[a, a + h]$ is such that

(i) f^{n-1} is continuous in the closed interval $[a, a + h]$,

(ii) f^{n-1} is derivable in the open interval $[a, a + h]$;

then there exists some θ satisfying $0 < \theta < 1$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h).$$

The quantity $R_n = \frac{h^n}{n!} f^n(a + \theta h)$ is called the Taylor's remainder after n terms.

The formal proof of the above theorem is beyond out syllabus (it will be given in the chapter on Mean-value theorem in Real Analysis) and as stated earlier it is derived by the application of Rolle's Theorem. However, we would like to point out at this stage that if the series for $f(a + h)$ is convergent, then the remainder $R_n \to 0$ as $n \to \infty$.

Thus the mention of the word 'convergent series' implies that R_n which stands for the remainder series after n terms tends to zero.

We now proceed to give an elementary proof of the theorem as embodied in the first statement.

Proof : Let us suppose that $f(x)$ possesses derivatives of every order in the closed interval $[a, a + h]$ and that $f(x + h)$ can be expanded in the form of a convergent series of positive integral powers of h in the following way:

$$f(x + h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n + \ldots \ldots (I)$$

where $A_0, A_1, A_2, A_3 \ldots A_n, \ldots$ are functions of x independent of h.

Here the main work is to determine the value of $A_0, A_1, A_2, A_3, \ldots A_n \ldots$ For this, we shall go on differentiating both the sides of the series (I) w.r.t. h and each time after differentiation we shall put $h = 0$.

Now, putting $h = 0$ in (I), we get $A_0 = f(x)$.

Again, differentiating both the sides of (I) w.r.t. h,

$$f'(x + h) = A_1 + A_2 . 2h + A_3 . 3h^2 + \ldots + A_n . nh^{n-1} + \ldots \ldots (II)$$

Now, putting $h = 0$ in (II), $A_1 = f'(x)$.

Again, differentiating both sides of (II) w.r.t. h, we get

$$f''(x + h) = A_2 . 2 + A_3 . 3 . 2h + \ldots + A_n . n(n-1)h^{n-2} + \ldots \ldots (III)$$

Now, putting $h = 0$ in (III), $2A_2 = f''(x)$. $\quad \therefore \quad A_2 = \frac{1}{2!} f''(x)$.

Again, differentiating both sides of (III) w.r.t. h, we get

$$f'''(x+h) = A_3 3.2 + \ldots + A_n n(n-1)(n-2)h^{n-3} + \ldots \quad \ldots(IV)$$

Now, putting $h = 0$ in (IV), $3.2.1\ A_3 = f'''(x)$. $\therefore \quad A_3 = \frac{1}{3!} f'''(x)$.

Proceeding in a similar manner, after taking the nth differential of $f(x+h)$, we get

$f^n(x+h) = A_n n! +$ terms containing powers of h.

Hence, putting $h = 0$,

$$A_n\, n! = f^n(x) \quad \therefore \quad A_n = \frac{1}{n!} f^n(x).$$

Now putting the values of $A_0, A_1, A_2, \ldots A_n \ldots$ in (I), we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots + \ldots + \frac{h^n}{n!} f^n(x) + \ldots$$

Hence proved.

The series is called **Taylor's Series** after the name of the English mathematician Brook Taylor (1685–1731).

Note : As pointed out earlier, the remainder after n terms of the series is given by

$$R_n = \frac{h^n}{n!} f^n(x + \theta h)$$

so that for the convergency of the series $R_n \to 0$.

2.3. Maclaurin's Theorem

According to Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots + \frac{h^n}{n!} f^n(x) + \ldots$$

Putting $x = 0$ in the above series

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + \ldots + \frac{h^n}{n!} f^n(0) + \ldots$$

Now, writing x in place of h, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

This is known as Maclaurin's series and this is frequently used in problems concerning expansion.

Note : We also use the notations y_0 for $f(0)$, $(y_1)_0$ for $f'(0)$, $(y_2)_0$ for $f''(0)$, $(y_3)_0$ for $f'''(0)$ ... and so on.

Independent proof of Maclaurin's Theorem

Statement : *If the function $f(x)$ possesses derivatives of every order in $[0, x]$ and can be expanded in a convergent series of positive integral powers of x, then that expansion is*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

Proof : We suppose that $f(x)$ can be expanded in the form of a convergent series of positive integral powers of x. Thus we suppose that

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots \quad \ldots(1)$$

where $A_0, A_1, A_2, A_3, \ldots$ are constants independent of x, which are to be determined.

Putting $x = 0$ in (1), $A_0 = f(0)$.

Now, differentiating (1) w.r.t. x,

$$f'(x) = A_1 + A_2 \cdot 2x + A_3 \cdot 3x^2 + \ldots \quad \ldots(2)$$

Putting $x = 0$ in (2), $A_1 = f'(0)$.

Againg, differentiating (2) w.r.t. x,

$$f''(x) = A_2.2 + A_3.3.2x + A_4.4.3x^2 + \ldots \quad \ldots(3)$$

Putting $x = 0$ in (3), we get $2A_2 = f''(0)$.

$$\Rightarrow \quad A_2 = \frac{1}{2!}f''(0).$$

Again, differentiating (3) w.r.t. x,

$$f'''(x) = A_3.3.2.1 + A_4.4.3.2x \ldots \quad \ldots(4)$$

Putting $x = 0$ in (4), we get $3!\, A_3 = f'''(0)$,

$$\Rightarrow \quad A_3 = \frac{1}{3!}f'''(0).$$

Similarly $A_4 = \frac{1}{4!}f^{iv}(0)$ ... etc.

Now, putting the values of $A_0, A_1, A_2, A_3, A_4, \ldots$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

Note : In this case,

$$R_n = \frac{x^n}{n!}f^n(\theta x) \quad \text{where} \quad 0 < \theta < 1.$$

The above expansion is called the expansion around $x = 0$.

If we want the expansion around $x = a$, we proceed as follows :

We have,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

Replacing h by $x - a$, we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

which is called expansion around $x = a$.

2.4. Failure of Taylor's Series and Maclaurin's Series

The expansion of any function $f(x)$ by Taylor's series will not be valid for the following values of x if

(i) $f(x)$ or any one of its derivatives is infinite

(ii) if the series of the expansion is not convergent.

We take up the following examples in support of the above points.

EXAMPLE 1. *Let* $f(x) = e^{\frac{1}{x}}$ *where* $x \neq 0$, *and* $f(0) = 0$.

SOLUTION : Then $f'(x) = -\frac{1}{x^2} \cdot e^{\frac{1}{x}} \quad \therefore \; f'(0) = \infty$ etc.

$\Rightarrow$ the differential coeff. of any order for the function $e^{\frac{1}{x}}$ is not finite.

Therefore the expansion of $e^{\frac{1}{x}}$ in ascending powers of x is not possible.

EXAMPLE 2. *Let* $f(x) = \sqrt{x}$.

SOLUTION : Here $f'(0), f''(0), f'''(0) \ldots$ are all infinite.

Hence the expansion of $\sqrt{x}$ in ascending powers of x is not possible.

EXAMPLE 3. *Let* $f(x) = \log x$.

SOLUTION : Here also, $f(0), f'(0), f''(0), \ldots$ all are infinite.

Hence the expansion of $\log x$ in ascending powers of x is not possible.

Note : The second condition of Taylor's series is that the series on the R.H.S. should be convergent. As pointed out earlier, if we denote the remainder series after the n th term by R_n [which we know is $= \frac{x^n}{n!} f^n(\theta x)$], then it is necessary that $R_n \to 0$ as $n \to \infty$.

2.5. Expansion of some well-known series by means of Maclaurin's Theorem.

Here we derive their expansions by following the elementary method.

I. Expansion of e^x :

Let $\quad f(x) = e^x \quad \therefore \quad f(0) = 1.$

Now differentiating successively, we get

$$f'(x) = e^x \quad \therefore \ f'(0) = 1$$

$$f''(x) = e^x \quad \therefore \ f''(0) = 1$$

$$f'''(x) = e^x \quad \therefore \ f'''(0) = 1$$

.....

Generally $f^n(x) = e^x \quad \therefore \ f^n(0) = 1.$

Hence from Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

We know from Analysis that the series on R.H.S. is convergent for all real x.

II. Expansion of $\log(1 + x)$: **(B.U. 66)**

Let $f(x) = \log(1 + x) \quad \therefore \ f(0) = \log 1 = 0.$

Now differentiating successively, we get

$$f'(x) = \frac{1}{1+x} \quad \therefore \ f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \quad \therefore \ f''(0) = -1$$

$$f'''(x) = \frac{(-1)(-2)}{(1+x)^3} \quad \therefore \ f'''(0) = 2$$

.....

$$\text{Generally } f^n(x) = \frac{(-1)(-2)(-3)\ldots(-n-1)}{(1+x)^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

$$\therefore \quad f^n(0) = (-1)^n(n-1)!.$$

Hence from Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \ \log(1+x) = 0 + x.(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}.2! + \ldots + \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

The series on R.H.S. is convergent when $-1 < x \leq 1$.

III. Expansion of $\sin x$: **(R.U. 67; P.U. 47S, 79)**

Let $f(x) = \sin x \quad \therefore \ f(0) = 0.$

Now differentiating successively, we get

$$f'(x) = \cos x \qquad \therefore \quad f'(0) = 1$$
$$f''(x) = -\sin x \qquad \therefore \quad f''(0) = 0$$
$$f'''(x) = -\cos x \qquad \therefore \quad f'''(0) = -1$$

.....

Generally, $f^n(x) = \sin\left(x + \frac{n\pi}{2}\right) \quad \therefore \quad f^n(0) = \sin\frac{n\pi}{2}$.

Hence from Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \quad \sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \ldots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \ldots$$

The sine–series is convergent for all real x.

IV. Expansion of $\cos x$: **(P.U. 81A)**

Let $f(x) = \cos x \qquad \therefore \quad f(0) = 1.$

Now, differentiating successively

$$f'(x) = -\sin x \qquad \therefore \quad f'(0) = 0$$
$$f''(x) = -\cos x \qquad \therefore \quad f''(0) = -1$$
$$f'''(x) = -\sin x \qquad \therefore \quad f'''(0) = 0$$

.....

Generally $f^n(x) = \cos\left(x + \frac{n\pi}{2}\right) \quad \therefore \quad f^n(0) = \cos\frac{n\pi}{2}$.

Hence from Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \quad \cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \ldots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \ldots$$

The cosine-series is convergent for all real x.

V. Expansion of $\tan x$: **(Kanpur 84)**

Let $y = f(x) = \tan x \qquad \therefore \quad f(0) = 0$

Differentiating, $y_1 = \sec^2 x \qquad \therefore \quad f'(0) = 1.$

We can write y_1 in the following way : $y_1 = 1 + \tan^2 x = 1 + y^2$.

Again differentiating

$$y_2 = 2yy_1 \qquad \therefore \quad (y_2)_0 = f''(0) = 0.$$

Again differentiating successively, we get

$$y_3 = 2(yy_2 + y_1^2) \qquad \therefore \ (y_3)_0 = 2.1 = 2$$

$$y_4 = 2\{(yy_3 + y_2y_1) + 2y_1y_2\}$$

$$= 2(yy_3 + 3y_1y_2) \qquad \therefore \ (y_4)_0 = 0$$

$$y_5 = 2\{(yy_4 + y_3y_1) + 3(y_1y_3 + y_2^2)\}$$

$\therefore \ (y_5)_0 = 2\{2 + 3(2)\} = 16$ etc.

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

or $\tan x = 0 + x + 0 + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \ldots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$

VI. Expansion of $\tan^{-1}x$.

Let $f(x) = \tan^{-1}x \qquad \therefore \ f(0) = 0$

$$f'(x) = \frac{1}{1 + x^2} \qquad \therefore \ f'(0) = 1 .$$

The nth derivative is given to be

$$f^n(x) = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta \text{ where } \tan\theta = \frac{1}{x} .$$

When $x = 0$, $\tan\theta = \infty$ so that $\theta = \frac{\pi}{2}$.

$$\therefore f^n(0) = (-1)^{n-1}(n-1)!\left(\sin\frac{\pi}{2}\right)^n \sin\frac{n\pi}{2}$$

$$= (-1)^{n-1}(n-1)!\sin\frac{n\pi}{2} .$$

Now, putting $n = 2, 3, 4, \ldots$, we get

$f''(0) = -\sin\pi = 0$

$$f'''(0) = +2!\sin\frac{3\pi}{2} = -2$$

$$f^4(0) = -3!\sin\frac{4\pi}{2} = 0$$

$$f^5(0) = +4!\sin\frac{5\pi}{2} = 4! \text{ etc.}$$

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

i.e. $\tan^{-1}x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(4!) + \ldots$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$$

VII. Expansion of $(1+x)^n$ when n is either negative or fractional.

Let $f(x) = (1+x)^n$ where n is either negative or fractional.

Then, f possesses derivatives of every order and so

$$f^m(x) = n(n-1)(n-2)\ldots(n-m+1)(1+x)^{n-m}$$

Now, $f(0) = 1$, $f'(0) = n$

$f''(0) = n(n-1)$, $f'''(0) = n(n-1)(n-2)$

.....

Hence from Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$= 1 + nx + \frac{x^2}{2!}\cdot n(n-1) + \frac{x^3}{3!}n(n-1)(n-2) + \ldots$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

We know from Analysis that the series on the R.H.S. is convergent if $|x| < 1$.

2.6. Summary

Method II. *Algebraical and Trigonometrical methods.*

Thus by the use of Maclaurin's series we obtain the following three series from Algebra and the following three series from Trigonometry which can be further used for the expansion of a given function :

(i) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \ldots$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

(vi) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

Apart from these, we have got another series from Trigonometry (whose expansion is given on page 63).

(vii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Worked out Examples

EXAMPLE 1. *Expand* $\log\sin(x+h)$ *in ascending powers of* h *as far as the term involving* h^3. **(B.U. 58A; P.U. 77A; M.U. 84)**

SOLUTION : Let $f(x) = \log\sin x$.

$$\therefore \quad f'(x) = \frac{1}{\sin x} \times \cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = -2\operatorname{cosec} x(-\operatorname{cosec} x\cot x) = 2\operatorname{cosec}^2 x\cot x.$$

Therefore according to Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

$$\therefore \log\sin(x+h) = \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}2\operatorname{cosec}^2 x\cot x + \ldots$$

EXAMPLE 2. *Prove by Taylor's Theorem that*

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha \cdot \frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - \ldots$$

where $x = \cot\alpha$. **(M.U. 86; Gorakhpur 85)**

SOLUTION : Let $f(x) = \tan^{-1}x$.

We know that the nth differential coeff. of $f(x)$ i.e.

$$f^n(x) = (-1)^{n-1}(n-1)!\sin^n\alpha\sin n\alpha, \text{ where } x = \cot\alpha.$$

Now putting $n = 1, 2, 3, \ldots$ successively, we get

$$f'(x) = \sin\alpha\sin\alpha,\ f''(x) = -\sin^2\alpha\ \sin 2\alpha;$$

$$f'''(x) = 2!\sin^3\alpha\sin 3\alpha \text{ etc.}$$

Hence, according to the Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

$$\text{or } \tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\sin\alpha - \frac{h^2}{2}\sin^2\alpha\sin 2\alpha$$

$$+\frac{h^3}{3!} \cdot 2!\sin^3\alpha\sin 3\alpha - \ldots.$$

$$= \tan^{-1}x + h\sin\alpha.\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - \ldots$$

EXAMPLE 3. *Expand* $\log\cos x$ *by Maclaurin's Theorem.* **(P.U. 49S, 56A, 68S, M.U. 65, 92)**

SOLUTION : Let $y = \log\cos x$.

Then $y_1 = \frac{1}{\cos x} \times (-\sin x) = -\tan x \quad \therefore (y_1)_0 = 0.$

Again differentiating

$$y_2 = -\sec^2 x = -(1 + \tan^2 x) = -(1 + y_1^{\,2}) \quad \therefore (y_2)_0 = -1.$$

Again, $y_3 = -2y_1y_2 \quad \therefore (y_3)_0 = 0.$

$$y_4 = -2(y_1y_3 + y_2^{\,2}) \quad \therefore (y_4)_0 = -2 \times 1 = -2.$$

$$y_5 = -2\{(y_1y_4 + y_3y_2) + 2y_2y_3\}$$

$$= -2y_1y_4 - 6y_2y_3 \quad \therefore (y_5)_0 = 0.$$

Again, $y_6 = -2(y_1y_5 + y_4y_2) - 6(y_2y_4 + y_3^{\,2})$

$\therefore (y_6)_0 = -2(2) - 6(2) = -16$ etc.

Hence from Maclaurin's Theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$= 0 + 0 + \frac{x^2}{2!}(-1) + \frac{x^4}{4!}(-2) + \frac{x^6}{6!}(-16) + \dots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \dots$$

Another method :

Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\therefore \log \cos x = \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$= \log\left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \dots\right)\right\}$$

$$= -\left(x^2 - \frac{x^4}{24} + \frac{x^6}{720} + \dots\right) - \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \dots\right)^2$$

$$- \frac{1}{3}\left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \dots\right)^3 + \dots$$

$$= -\left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \dots\right) - \frac{1}{2}\left(\frac{x^4}{4} - 2.\frac{x^2}{2}\cdot\frac{x^4}{24} + \dots\right)$$

$$- \frac{1}{3}\left(\frac{x^6}{8} + \dots\right) + \dots$$

$$= -\frac{x^2}{2} + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + x^6\left(-\frac{1}{720} + \frac{1}{48} - \frac{1}{24}\right) + \dots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \dots$$

Note : $\log \sec x = \log \frac{1}{\cos x} = -\log \cos x$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$$ **(M.U. 71A)**

EXAMPLE 4. *Apply Maclaurin's Theorem to obtain the expansion of* $\sec x$. **(B.U. 64A; R.U. 63A, 71A; P.U. 42, 48A; M.U. 74A, 79)**

SOLUTION : Let $y = f(x) = \sec x$ $\quad\therefore\ f(0) = 1$.

Then $\quad y_1 = \sec x \tan x \quad\therefore\ f'(0) = 0.$

Differentiating this, we get

$$y_2 = \sec^3 x + \tan x.\sec x.\tan x = \sec^3 x + \sec x.\tan^2 x$$

$$= \sec^3 x + \sec x(\sec^2 x - 1) = 2\sec^3 x - \sec x$$

$$= 2y^3 - y \qquad \dots(1)$$

$\therefore\quad (y_2)_0 = 2 - 1 = 1.$

Again, differentiating (1), $y_3 = 2.3y^2.y_1 - y_1 = 6y^2y_1 - y_1$ $\qquad\dots(2)$

$\therefore\quad (y_3)_0 = 0.$

Again, differentiating (2), $y_4 = 6.(y^2y_2 + y_1.2y.y_1) - y_2$

$\therefore\quad (y_4)_0 = 6(1 + 0) - 1 = 5$ etc.

Hence according to Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

or $\quad \sec x = 1 + x.0 + \frac{x^2}{2!}(1) + \frac{x^4}{4!}(5) + \dots$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots$$

Another Method :

We have, $\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^{-1} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)\right\}^{-1}$$

$$= (1 - y)^{-1} \text{ where } y = \frac{x^2}{2} - \frac{x^4}{24} + \ldots$$

$$= 1 + y + y^2 + y^3 + \ldots$$

Now putting the value of y, this is

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right) + \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)^2 + \left(x^2 - \frac{x^4}{24} + \ldots\right)^3 + \ldots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right) + \left(\frac{x^4}{4} + \ldots\right) + \ldots$$

$$= 1 + \frac{x^2}{2} + x^4\left(\frac{1}{4} - \frac{1}{24}\right) + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$$

EXAMPLE 5. *Expand* $\tan x$ *in powers of* x *as far as* x^5. **(P.U. 53S)**

SOLUTION : The expansion of $\tan x$ by using Maclaurin's series has been found out on page 63. However, we emply the trigonometrical series of $\sin x$ and $\cos x$ to derive the expansion.

We know from Trigonometry that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left\{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)\right\}^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left\{1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)^2 + \ldots\right\}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left\{1 + \frac{x^2}{2!} + x^4\left(-\frac{1}{4!} + \frac{1}{4}\right) + \ldots\right\}$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= \left(x + \frac{x^3}{2} + \frac{5}{24}x^5 + \ldots\right) + \left(-\frac{x^3}{6} - \frac{x^5}{12} - \ldots\right) + \left(\frac{x^5}{120} + \ldots\right)$$

$$= x + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + x^5\left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right) + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

Note : This result should be remembered as a formula.

EXAMPLE 6. *Apply Maclaurin's Theorem to obtain the expansion of* $\log(1 + \tan x)$. **(P.U. 53A; Bhag. 62A; M. U. 68A, 72A)**

SOLUTION : Let $y = f(x) = \log(1 + \tan x)$.

$\therefore\ (y)_0 = \log 1 = 0$, i.e. $y = 0$ when $x = 0$.

From the given relation, $e^y = 1 + \tan x$

$$\therefore \qquad e^y.y_1 = \sec^2 x \qquad \ldots(1)$$

$$\therefore \quad e^0(y_1)_0 = \sec^2 0. \Rightarrow (y_1)_0 = 1.$$

Again from (1), $e^y.y_2 + y_1 e^y.y_1 = 2\sec x.\sec x\tan x$

$$\Rightarrow \quad e^y(y_2 + y_1{}^2) = 2\sec^2 x\tan x \qquad \ldots(2)$$

$$\therefore \quad 1\{(y_2)_0 + 1\} = 0 \quad \therefore (y_2)_0 = -1.$$

Again from (2),

$$e^y(y_3 + 2y_1y_2) + (y_2 + y_1{}^2)e^y.y_1$$

$$= 2\{\sec^2 x \sec^2 x + \tan x.2\sec^2 x \tan x\}$$

$$\therefore \qquad (y_3)_0 + 2.1(-1) + (-1 + 1).1 = 2(1)$$

$$\Rightarrow \qquad (y_3)_0 - 2 = 2 \qquad \therefore (y_3)_0 = 4.$$

Hence from Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

i.e. $\log(1 + \tan x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(4) + \ldots$

$$= x - \frac{x^2}{2} + \frac{2}{3}x^3 + \ldots$$

Another Method :

We know that $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Therefore $\log(1 + \tan x) = \log\left\{1 + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right)\right\}$

$$= \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right) - \frac{1}{2}\left(x + \frac{x^3}{3} + \ldots\right)^2 + \frac{1}{3}\left(x + \frac{x^3}{3} + \ldots\right)^3 - \frac{1}{4}\left(x + \frac{x^3}{3} + \ldots\right)^4 + \ldots$$

$$= \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right) - \frac{1}{2}\left(x^2 + 2x \cdot \frac{x^3}{3} + \ldots\right) + \frac{1}{3}(x^3 + \ldots) - \frac{1}{4}(x^4 + \ldots) + \ldots$$

$$= x - \frac{x^2}{2} + x^3\left(\frac{1}{3} + \frac{1}{3}\right) + x^4\left(-\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

$$= x - \frac{x^2}{2} + \frac{2}{3}x^3 - \frac{7}{12}x^4 + \ldots$$

EXAMPLE 7. *Expand* $\log(1 + \sin x)$ *in ascending powers of* x *by Maclaurin's Theorem up to* x^4. **(P.U. 56S, 59A; B.U. 53A; Bhag. 64A, 65S, 87; M.U. 72S; R.U. 73A, 83; Kanpur 86)**

SOLUTION: Let $y = f(x) = \log(1 + \sin x)$.

$\therefore \quad f(0) = \log 1 = 0.$

We write $y = \log(1 + \sin x)$ in the following manner :—

$$y = \log\left\{1 + \cos\left(\frac{\pi}{2} - x\right)\right\}$$

Then $y_1 = \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)} = \dfrac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

$\therefore \quad f'(0) = \tan\dfrac{\pi}{4} = 1.$

Again $y_2 = -\dfrac{1}{2}\sec^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$

$$= -\frac{1}{2}\left\{1 + \tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = -\frac{1}{2}(1 + y_1^{\,2})$$

$\therefore \quad f''(0) = -\dfrac{1}{2}(1 + 1) = -1.$

Again $y_3 = -\dfrac{1}{2}(2y_1y_2) \qquad \therefore \quad f'''(0) = -\{1 \times (-1)\} = 1.$

Again $y_4 = -(y_1 y_3 + y_2^2)$ $\quad \therefore \quad f^{iv}(0) = -(1 + 1) = -2.$

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$

$$= 0 + x + \frac{x^2}{2!}(-1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-2) + \dots$$

i.e. $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

Note : By putting the given equation in the form $e^y = 1 + \sin x$, we could have evaluated $f(0), f'(0), \dots$ etc.

Another Method :

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$\therefore \quad \log(1 + \sin x) = \log\left\{1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\right\}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2$$

$$+ \frac{1}{3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^4 + \dots$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)^2$$

$$+ \frac{1}{3}\left(x - \frac{x^3}{6} + \dots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{6} + \dots\right)^4 + \dots$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) - \frac{1}{2}\left(x^2 - 2.x.\frac{x^3}{6} + \dots\right)$$

$$+ \frac{1}{3}(x^3 - \dots) - \frac{1}{4}(x^4 - \dots) + \dots$$

$$= x - \frac{x^2}{2} + x^3\left(-\frac{1}{6} + \frac{1}{3}\right) + x^4\left(\frac{1}{6} - \frac{1}{4}\right) + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$$

EXAMPLE 8. *Expand* $\log(1 + \sin^2 x)$ *by using Maclaurin's Theorem.*

SOLUTION : Let $y = \log(1 + \sin^2 x)$ $\quad \therefore \quad f(0) = 0.$

Then $\quad e^y = 1 + \sin^2 x.$

Therefore differentiating,

$e^y . y_1 = 2\sin x \cos x = \sin 2x \quad \therefore \quad (y_1)_0 = 0.$

Again, $e^y.y_2 + y_1{}^2e^y = 2\cos 2x$

$\therefore \quad (y_2)_0 + 0 = 2 \qquad \therefore \quad (y_2)_0 = 2.$

Again, $(e^y.y_3 + y_2e^y.y_1) + (y_1{}^3e^y + e^y.2y_1y_2) = -4\sin 2x$ ..(1)

$\therefore \quad (y_3)_0 + 0 + (0 + 0) = 0 \qquad \therefore \quad (y_3)_0 = 0.$

From (1), $e^y.y_3 + 3e^y.y_1y_2 + e^y.y_1{}^3 = -4\sin 2x = -4.e^y.y_1.$

$\Rightarrow \quad y_3 + 3y_1y_2 + y_1{}^3 = -4y_1.$

Again differentiating,

$$y_4 + 3(y_1y_3 + y_2{}^2) + 3y_1{}^2y_2 = -4y_2$$

$\therefore \quad (y_4)_0 + 3(4) + 0 = -4.2 \qquad \therefore \quad (y_4)_0 = -20$ etc.

Hence from Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

i.e. $\log(1 + \sin^2 x) = 0 + x(0) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-20) + \ldots$

$$= x^2 - \frac{5}{6}x^4 + \ldots$$

EXAMPLE 9. *Expand by Maclaurin's Theorem* $\log(1 + e^x)$ *as far as the term* x^3. **(P.U. 44A, 46S, 62S, 68A; 70A; R.U. 76; M.U. 87, 90)**

SOLUTION : Let $y = \log(1 + e^x) \qquad \therefore \quad (y)_0 = \log 2:$

Then $y_1 = \frac{e^x}{1 + e^x} = 1 - \frac{1}{1 + e^x} \qquad \therefore \quad (y_1)_0 = \frac{1}{2},$

$$y_2 = \frac{e^x}{(1 + e^x)^2} = \frac{e^x}{1 + e^x} \cdot \frac{1}{1 + e^x} = y_1 \cdot \frac{1}{1 + e^x} = y_1(1 - y_1)$$

$$= y_1 - y_1{}^2.$$

$\therefore \quad (y_2)_0 = (y_1)_0 - |(y_1)_0|^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

$$y_3 = y_2 - 2y_1y_2 \qquad \therefore \quad (y_3)_0 = \frac{1}{4} - 2.\frac{1}{2}.\frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0.$$

$$y_4 = y_3 - 2(y_1y_3 + y_2{}^2)$$

$\therefore \quad (y_4)_0 = 0 - 2\left(\frac{1}{2} \cdot 0 + \frac{1}{16}\right) = -\frac{1}{8}$ etc.

Hence, according to the Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

i.e. $\log(1 + e^x) = \log 2 + \frac{1}{2}x + \frac{1}{4}\cdot\frac{x^2}{2!} - \frac{1}{8}\cdot\frac{x^4}{4!} + \cdots$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

Note : Differentiating the expansion of $\log(1 + e^x)$, we get

$$\frac{e^x}{1 + e^x} = \frac{1}{2} + \frac{2x}{8} - \frac{4x^3}{192} + \cdots$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

Another Method :

We have, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Therefore $\log(1 + e^x) = \log\left\{2 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\right\}$

$$= \log\left[2\left\{1 + \frac{1}{2}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\right\}\right]$$

$$= \log 2 + \log\left\{1 + \frac{1}{2}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\right\}$$

$$= \log 2 + \frac{1}{2}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \frac{1}{2}\cdot\frac{1}{4}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$+ \frac{1}{3}\cdot\frac{1}{8}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 - \frac{1}{4}\frac{1}{16}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^4 + \cdots$$

$$= \log 2 + \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) - \frac{1}{8}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)^2$$

$$+ \frac{1}{24}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)^3 - \frac{1}{64}\left(x + \frac{x^2}{2} + \cdots\right)^4 + \cdots$$

$$= \log 2 + \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$

$$- \frac{1}{8}\left(x^2 + 2x\cdot\frac{x^2}{2} + \frac{x^4}{4} + 2x\cdot\frac{x^3}{6} + \cdots\right)$$

$$+ \frac{1}{24}\left(x^3 + 3x\cdot\frac{x^2}{2} + \cdots\right) - \frac{1}{64}(x^4 + \ldots) + \cdots$$

$$= \log 2 + \frac{x}{2} + x^2\left(\frac{1}{4} - \frac{1}{8}\right) + x^3\left(\frac{1}{12} - \frac{1}{8} + \frac{1}{24}\right)$$
$$+ x^4\left(\frac{1}{48} - \frac{1}{32} - \frac{1}{24} + \frac{1}{16} - \frac{1}{64}\right) + \ldots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$$

EXAMPLE 10. *Expand* $\frac{e^x}{1+e^x}$ *by Maclaurin's Theorem.* **(M.U. 67S)**

SOLUTION : Let $y = f(x) = \frac{e^x}{1+e^x}$ $\therefore$ $f(0) = \frac{1}{2}$

As in Ex. 8, we can write $f(x)$ as

$$y = 1 - \frac{1}{1+e^x}$$

$$\therefore \quad y_1 = \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x} \cdot \frac{1}{1+e^x}$$

$$= y \cdot \frac{1}{1+e^x} = y(1-y) = y - y^2.$$

$$\therefore \quad (y_1)_0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Like the previous exercise, $(y_2)_0 = 0$, $(y_3)_0 = -\frac{1}{8}$ etc.

$\therefore$ According to Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(-\frac{1}{8}\right) +$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots$$

EXAMPLE 11. *Expand* $e^{\sin x}$ *by Maclaurin's Theorem as far as the term involving* x^4. **(R.U. 64A, 68A, 70S; B.U. 55S, 57A, 71A; P.U. 40A, 50S, 71A, Bhag. 85; M.U. 91; Gorakhpur 86)**

SOLUTION : Let $y = e^{\sin x}$ $\therefore$ $f(0) = e^0 = 1$.

Then differentiating,

$$y_1 = e^{\sin x}\cos x = y\cos x \quad \therefore \quad f'(0) = 1.$$

Again $y_2 = y(-\sin x) + \cos x.y_1$ $\therefore$ $(y_2)_0 = 0 + 1.1 = 1$.

Again $y_3 = -(y\cos x + \sin x.y_1) + \{\cos x.y_2 + y_1(-\sin x)\}$

$$= y_2\cos x - 2y_1\sin x - y\cos x$$

$\therefore \quad (y_3)_0 = 1 - 0 - 1 = 0.$

Again

$y_4 = \{ y_2(-\sin x) + \cos x.y_3\} - 2(y_1\cos x + \sin x.y_2\} - \{y(-\sin x) + \cos x.y_1\}$

$\therefore \quad (y_4)_0 = 0 - 2(1) - (1) = -3$ etc.

Hence according to Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \cdots$$

i.e. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

Another Method :

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$\therefore \quad e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \cdots$$

$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2$$

$$+ \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3 + \frac{1}{4!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^4 + \cdots$$

$$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + \cdots\right)^3$$

$$+ \frac{1}{24}\left(x - \frac{x^3}{6} + \cdots\right)^4 + \cdots$$

$$= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x^2 - 2x.\frac{x^3}{6} + \cdots\right)$$

$$+ \frac{1}{6}(x^3 - \ldots) + \frac{1}{24}(x^4 - \ldots) + \cdots$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{6} + \frac{1}{6}\right) + x^4\left(-\frac{1}{6} + \frac{1}{24}\right) + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

EXAMPLE 12. *Expand* $\frac{e^x}{\cos x}$ $(= e^x \sec x)$ *by Maclaurin's Theorem.*

(B. U. 60A; M. U. 63A, 85; P. U. 65

SOLUTION : Let $y = f(x) = e^x \sec x \quad \therefore \quad f(0) = 1.$

Then $y_1 = e^x \sec x.\tan x + \sec x.e^x = y + y\tan x$

$= y(1 + \tan x)$ $\quad \therefore \quad (y_1)_0 = 1.$

Again, $y_2 = y(\sec^2 x) + (1 + \tan x)y_1 \quad \therefore \quad (y_2)_0 = 1 + 1 = 2.$

Again, $y_3 = y.2\sec^2 x\tan x + 2\sec^2 x.y_1 + (1+ \tan x)y_2$

$\therefore \quad (y_3)_0 = 0 + 2 + 2 = 4.$

We can write y_3 as follows :

$$y_3 = 2y(1 + \tan^2 x)\tan x + 2\sec^2 x.y_1 + (1 + \tan x).y_2$$

$$= 2y\tan x + 2y\tan^3 x + 2\sec^2 x.y_1 + (1 + \tan x)y_2$$

$$\therefore \quad y_4 = 2\{y\sec^2 x + \tan x.y_1\} + 2(y.3\tan^2 x.\sec^2 x + \tan^3 x.y_1\}$$

$$+ 2\{\sec^2 x.y_2 + y_1.2\sec^2 x\tan x\} + \{1 + \tan x)y_3 + y_2\sec^2 x\}$$

$\therefore \quad (y_4)_0 = 2(1) + 2(0) + 2(2) + (4 + 2) = 2 + 4 + 6 = 12.$

Hence according to Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$= 1 + x + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(4) + \frac{x^4}{4!}(12) + \ldots$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \ldots$$

Alternative Method :

First, we find out the expansion of $\frac{1}{\cos x} = \sec x.$

We have found out in Ex. 4 that

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$$

Therefore $\quad \frac{e^x}{\cos x} = e^x \sec x$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right) + \left(x + \frac{x^3}{2} + \ldots\right) + \left(\frac{x^2}{2} + \frac{x^4}{4} + \ldots\right)$$

$$+ \left(\frac{x^3}{6} + \ldots\right) + \left(\frac{x^4}{24} + \ldots\right)$$

$$= 1 + x + \left(\frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{2} + \frac{x^3}{6}\right) + \left(\frac{5}{24}x^4 + \frac{x^4}{4} + \frac{x^4}{24}\right) + \ldots$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \ldots$$

EXAMPLE 13. *Expand $e^x\sin^2 x$ by Maclaurin's Theorem up to x^4.* **(P.U. 63A)**

SOLUTION : Let $y = f(x) = e^x\sin^2 x \quad \therefore \; f(0) = 0$

$$\therefore \; y_1 = e^x.2\sin x\cos x + \sin^2 x e^x \qquad ...(1)$$

$\therefore \; f'(0) = 0.$

From (1), $y_1 = y + e^x\sin 2x.$

Now differentiating successively,

$$y_2 = y_1 + e^x.2\cos 2x + \sin 2x.e^x = y_1 + e^x.2\cos 2x + (y_1 - y)$$

$$= y_1 - y + 2e^x\cos 2x.$$

$$\therefore \quad (y_2)_0 = 0 - 0 + 2.1 = 2.$$

Again, $y_3 = 2y_2 - y_1 + 2\{e^x(-\sin 2x)2 + \cos 2x.e^x\}$

$$= 2y_2 - y_1 + 2\{-2e^x\sin 2x + e^x\cos 2x\}$$

$$= 2y_2 - y_1 + 2\{-2(y_1 - y) + \frac{1}{2}(y_2 - 2y_1 + y)\}$$

$$= 2y_2 - y_1 + 2.\frac{1}{2}\{-4(y_1 - y) + (y_2 - 2y_1 + y)\}$$

$$= 2y_2 - y_1 + \{-4y_1 + 4y + y_2 - 2y_1 + y\}$$

$$= 2y_2 - y_1 + (y_2 - 6y_1 + 5y) = 3y_2 - 7y_1 + 5y.$$

$$\therefore \quad (y_3)_0 = 3.2 - 0 + 0 = 6.$$

Again, $y_4 = 3y_3 - 7y_2 + 3y_1$

$$\therefore \quad (y_4)_0 = 3.6 - 7.2 + 0 = 18 - 14 = 4 \text{ etc.}$$

Now from Maclaurins Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\text{i.e. } e^x\sin^2 x = 0 + 0 + \frac{x^2}{2!}.2 + \frac{x^3}{3!}.6 + \frac{x^4}{4!}.4 + \ldots$$

$$= x^2 + x^3 + \frac{x^4}{6} + \ldots$$

EXAMPLE 14. *Obtain the expansion of $e^{ax}\cos bx$ in ascending powers of x.*

(P.U. 45A, 63A, 66A; R.U. 64S, 65S; M.U. 70A)

SOLUTION : First Method : From the formula

$$e^{ax}\cos bx = \left\{1 + (ax) + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots\right\} \times \left\{1 - \frac{(bx)^2}{2!} + \frac{(bx)^4}{4!} - \ldots\right\}$$

$$= \left\{1 - \frac{(bx)^2}{2!} + \frac{(bx)^4}{4!} - \ldots\right\} + \left\{(ax) - \frac{(ax)(bx)^2}{2!} + \ldots\right\}$$

$$+ \left\{\frac{(ax)^2}{2!} - \frac{(ax)^2}{2!} \cdot \frac{(bx)^2}{2!} + \ldots\right\} + \left\{\frac{(ax)^3}{3!} + \ldots\right\} + \left\{\frac{(ax)^4}{4!} + \ldots\right\} + \ldots$$

$$= 1 + ax + x^2\left(-\frac{b^2}{2} + \frac{a^2}{2}\right) + x^3\left\{-\frac{ab^2}{2} + \frac{a^3}{3!}\right\}$$

$$+ x^4\left\{\frac{b^4}{4!} - \frac{a^2}{2!} \cdot \frac{b^2}{2!} + \frac{a^4}{4!}\right\} + \ldots$$

$$= 1 + ax + \frac{a^2 - b^2}{2}x^2 + \frac{a^3 - 3ab^2}{6}x^3 + \frac{a^4 - 6a^2b^2 + b^4}{24}x^4 + \ldots$$

Second Method :

Let $y = f(x) = e^{ax}\cos bx \quad \therefore \quad f(0) = 1.$

$f'(x) = e^{ax}(-\sin bx) \times b + \cos bx.e^{ax}.a = a.e^{ax}\cos bx - be^{ax}\sin bx$

$\Rightarrow \quad y_1 = a.y - be^{ax}\sin bx \qquad \ldots(1)$

$\therefore \quad f'(0) = a.1 - 0 = a.$

Again differentiating (1),

$$y_2 = ay_1 - b\{e^{ax}\cos bx.b + \sin bx.e^{ax}.a\}$$

$$= ay_1 - b\left\{by + a\frac{ay - y_1}{b}\right\}; \text{ from (1)}$$

$$= ay_1 - b^2y - a(ay - y_1) = ay_1 - b^2y - a^2y + ay_1$$

$$= 2ay_1 - (a^2 + b^2)y. \qquad \ldots(2)$$

$\therefore \quad f''(0) = 2a.a - (a^2 + b^2).1 = 2a^2 - a^2 - b^2 = (a^2 - b^2).$

Again from (2),

$$y_3 = 2ay_2 - (a^2 + b^2)y_1 \qquad \ldots(3)$$

$\therefore \quad f'''(0) = 2a(a^2 - b^2) - (a^2 + b^2).a = 2a^3 - 2ab^2 - a^3 - ab^2$

$= a^3 - 3ab^2.$

Again from (3),

$$y_4 = 2ay_3 - (a^2 + b^2)y_2.$$

$\therefore \quad f^{iv}(0) = 2a(a^3 - 3ab^2) - (a^2 + b^2)(a^2 - b^2)$

$= 2a^4 - 6a^2b^2 - (a^4 - b^4) = a^4 - 6a^2b^2 + b^4$ etc.

Hence from Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$$

i.e. $e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2}x^2 + \frac{a^3 - 3ab^2}{6}x^3$

$$+ \frac{(a^4 - 6a^2b^2 + b^4)}{24}x^4 + \ldots$$

Cor. Putting $a = \cos\alpha$ and $b = \sin\alpha$ in the above expansion, we get

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots$$

EXAMPLE 15. *Obtain by Maclaurin's Theorem the first four terms of the expansion of $e^{x\cos x}$ in ascending powers of x.* **(P.U. 61A; B.U. 61A; Mithila 78A)**

SOLUTION : Let $y = f(x) = e^{x\cos x}$ $\therefore$ $f(0) = 1$.

Now taking logarithm, we have $\log y = x\cos x$.

Now, differentiating this, we have

$$\frac{1}{y} \cdot y_1 = x(-\sin x) + \cos x.1 \quad \therefore \quad y_1 = y(\cos x - x\sin x)$$

$\therefore$ $f'(0) = 1(1 - 0) = 1$.

Again differentiating, we have

$$y_2 = y\{-\sin x - (x\cos x + \sin x)\} + (\cos x - x\sin x)y_1$$

$$= -y(2\sin x + x\cos x) + (\cos x - x\sin x)y_1$$

$\therefore$ $f''(0) = 1(0) + (1)1 = 1$.

Again differentiating third time,

$$y_3 = -[y\{2\cos x + x(-\sin x) + \cos x\} + (2\sin x + x\cos x)y_1]$$

$$+ \{(\cos x - \sin x)y_2 + y_1(-2\sin x - x\cos x)\}$$

$\therefore$ $f'''(0) = -\{1(3) + 0\} + \{(1)1 + 1(0)\} = -3 + 1 = -2$.

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

i.e. $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$.

Second Method : Without the application of Maclaurin's theorem.

For this, we use the following formula :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Here putting $x\cos x$ for x, we get

$$e^{x\cos x} = 1 + x\cos x + \frac{x^2}{2!}\cos^2 x + \frac{x^3}{3!}\cos^3 x + \ldots$$

$$= 1 + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^2$$

$$+ \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^3 + \ldots ;$$

for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$$= 1 + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \frac{x^2}{2}(1 - x^2 + \ldots) + \frac{x^3}{6}(1 - \ldots) + \ldots$$

$$= 1 + \left(x - \frac{x^3}{2} + \ldots\right) + \left(\frac{x^2}{2} - \ldots\right) + \frac{x^3}{6}(1 - \ldots)$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{2}\right) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$$

EXAMPLE 16. *Expand* $\frac{x}{e^x - 1}$ *as far as* x^4. **(P.U. 55S)**

SOLUTION : Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$$\therefore \quad e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Hence $\frac{x}{e^x - 1} = \frac{x}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots}$

$$= \frac{1}{1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \ldots}$$

$$= \left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \ldots\right)^{-1}$$

$$= \left\{1 + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \ldots\right)\right\}^{-1}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \ldots\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \ldots\right)^2$$

$$- \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \ldots\right)^3 + \left(\frac{x}{2} + \frac{x^2}{6} + \ldots\right)^4 + .$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \dots\right)$$
$$+ \left(\frac{x^2}{4} + \frac{x^4}{36} + 2\frac{x}{2}\cdot\frac{x^2}{6} + 2\cdot\frac{x}{2}\cdot\frac{x^3}{24} + \dots\right)$$
$$- \left(\frac{x^3}{8} + 3\cdot\frac{x^2}{4}\cdot\frac{x^2}{6} + \dots\right) + \left(\frac{x^4}{16} + \dots\right) + \dots$$
$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4}\right) + x^3\left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8}\right)$$
$$+ x^4\left(-\frac{1}{120} + \frac{1}{36} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16}\right) + \dots$$
$$= 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots$$

EXAMPLE 17. *Expand* $\log(x \cot x)$ *in powers of* x *up to* x^4. **(P.U. 53S)**

SOLUTION : We have,

$$\log(x\cot x) = \log\frac{x}{\tan x} = -\log\frac{\tan x}{x}$$
$$= -\log\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots.}{x}$$
$$= -\log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right)$$
$$= -\log\left\{1 + \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right)\right\}$$
$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right)^2\right]$$
$$+ \frac{1}{3}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right)^3 + \dots.$$
$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \dots\right) - \frac{1}{2}\left(\frac{x^4}{9} + \dots\right) + \dots\right]$$
$$= -\left[\frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \dots\right]$$
$$= -\left[\frac{x^2}{3} + x^4\cdot\frac{12-5}{90} + \dots\right]$$
$$= -\frac{x^2}{3} - \frac{7}{90}x^4 + \dots$$

EXAMPLE 18. *Expand $e^x \cos x$ as far as terms containing x^4* **(Kanpur 85)**

SOLUTION : We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

and $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\therefore \quad e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) \times \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$$

$$= \left\{1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right\} + \left\{x - \frac{x^3}{2} + \ldots\right\} + \left\{\frac{x^2}{2} - \frac{x^4}{4} + \ldots\right\}$$

$$+ \left\{\frac{x^3}{6} - \ldots\right\} + \left\{\frac{x^4}{24} - \ldots\right\}$$

$$= 1 + x + x^2\left(-\frac{1}{2} + \frac{1}{2}\right) + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right) + \ldots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots$$

EXAMPLE 19. *Find the expansion of $\sin(e^x - 1)$ in powers of x up to x^4.*
(B.U. 62S; P.U. 69A; Gorakhpur 82, 88)

SOLUTION : We have,

$$\sin(e^x - 1) = \sin\left\{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1\right\}$$

$$= \sin\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)^3 + \ldots;$$

from the expansion of $\sin x$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \frac{1}{3!}\left(x^3 + 3x^2 \cdot \frac{x^2}{2!} \ldots\right) + \ldots$$

$$\{\because \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\}$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) - \frac{1}{6}\left(x^3 + \frac{3}{2}x^4 + \ldots\right) + \ldots$$

$$= x + \frac{x^2}{2} + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \ldots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots$$

EXAMPLE 20. *Expand* $f(x) = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1}$ *in powers of* x.

SOLUTION : Changing the sign of x, we have

$$f(-x) = \frac{-x}{2} \cdot \frac{e^{-x} + 1}{e^{-x} - 1} = -\frac{x}{2} \cdot \frac{1 + e^x}{1 - e^x}$$

$$= \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = f(x).$$

Thus $f(x)$ is an even function of x.

By Maclaurin's Theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + \ldots \quad \ldots(1)$$

$$\Rightarrow \quad f(-x) = f(0) - xf'(0) + \frac{x^2}{2!} f''(0) - \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) - \ldots \quad \ldots(2)$$

Adding (1) and (2), we get {since $f(-x) = f(x)$}

$$2f(x) = 2\left[f(0) + \frac{x^2}{2!} f''(0) + \frac{x^4}{4!} f^4(0) + \ldots\right]$$

i.e. $f(x) = f(0) + \frac{x^2}{2!} f''(0) + \frac{x^4}{4!} f^4(0) + \ldots$ ...(3)

Subtracting (2) from (1), we get.

$$0 = 2\left[xf'(0) + \frac{x^3}{3!} f'''(0) + \ldots\right]$$

This $\Rightarrow f'(0) = 0, \quad f'''(0) = 0, \quad f^5(0) = 0, \ldots$ ...(4)

Given $f(x) = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1}$.

$$\therefore \quad f(0) = \lim_{x \to 0} \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = \frac{2}{2} \lim_{x \to 0} \frac{x}{e^x - 1} = 1 \times 1 = 1.$$

From (2), $(e^x - 1)f(x) = \frac{x}{2} e^x + \frac{x}{2}$.

$$\Rightarrow \quad e^x f(x) = f(x) + \frac{x}{2} e^x + \frac{x}{2}.$$

Now differentiating successively, we get

$$e^x f'(x) + f(x)e^x = f'(x) + \frac{1}{2} xe^x + \frac{1}{2} e^x + \frac{1}{2} \quad \ldots(A)$$

$$\{e^x f''(x) + f'(x)e^x\} + \{f(x)e^x + e^x f'(x)\} = f''(x) + \frac{1}{2} xe^x + \frac{1}{2} e^x + \frac{1}{2} e^x$$

$$\Rightarrow \quad e^x\{f''(x) + 2f'(x) + f(x)\} = f''(x) + \frac{e^x}{2}(x + 2) \quad \ldots(B)$$

Similarly,

$$e^x\{f'''(x) + 3f''(x) + 3f'(x) + f(x)\} = f'''(x) + \frac{e^x}{2}(x+3) \quad \ldots(C)$$

$$e^x\{f^4(x) + 4f'''(x) + 6f''(x) + 4f'(x) + f(x)\} = f^4(x) + \frac{e^x}{2}(x+4) \quad \ldots(D)$$

$$e^x\{f^5(x) + 5f^4(x) + 10f'''(x) + 10f''(x) + 5f'(x) + f(x)\}$$

$$= f^5(x) + \frac{e^x}{2}(x+5) \quad \ldots(E)$$

Putting $x = 0$ in (A), (B), (C), (D), (E), ... respectively, we get,

$$f'(0) + f(0) = f'(0) + 0 + 1 \Rightarrow f(0) = 1;$$

$$f''(0) + 2f'(0) + f(0) = f''(0) + \frac{1}{2} \cdot 2 \Rightarrow f'(0) = 0$$

$$f'''(0) + 3f''(0) + 3f'(0) + f(0) = f'''(0) + \frac{1}{2} \cdot 3$$

$$\Rightarrow \quad 3f''(0) + 0 + 1 = \frac{3}{2} \Rightarrow 3f''(0) = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow f''(0) = \frac{1}{6}$$

$$f^4(0) + 4f'''(0) + 6f''(0) + 4f'(0) + f(0) = f^4(0) + \frac{1}{2} \cdot 4.$$

$$\Rightarrow \quad 4 \times f'''(0) + 6\left(\frac{1}{6}\right) + 4 \times 0 + 1 = 2 \Rightarrow f'''(0) = 0;$$

$$f^5(0) + 5f^4(0) + 10f'''(0) + 10f''(0) + 5f'(0) + f(0) = f^5(0) + \frac{1}{2} \cdot 5$$

$$\Rightarrow \quad 5f^4(0) + 0 + 10 \times \frac{1}{6} + 0 + 1 = \frac{5}{2} \Rightarrow 5f^4(0) = \frac{5}{2} - 1 - \frac{5}{3} = -\frac{1}{6}$$

$$\therefore \quad f^4(0) = -\frac{1}{30} \ldots$$

Hence from (3)

$$\frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = 1 + \frac{x^2}{2!} \cdot \frac{1}{6} + \frac{x^4}{4!}\left(-\frac{1}{30}\right) + \ldots$$

$$= 1 + \frac{1}{6} \cdot \frac{x^2}{2!} - \frac{1}{30} \cdot \frac{x^4}{4!} + \ldots$$

Note : The numbers $f''(0), f^4(0), f^6(0), \ldots$ are called Bernoulli's numbers and they are denoted by $B_1, B_3, B_5, \ldots$ where $B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$ etc.

Thus the above expansion can be written as

$$\frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = 1 + B_1\frac{x^2}{2!} - B_3\frac{x^4}{4!} + B_5\frac{x^6}{6!} + \ldots$$

Cor. 1. Expansion of $x\cot hx$.

We have, $x\cot hx = x\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{2x}{2} \cdot \dfrac{e^{2x}+1}{e^{2x}-1}$.

Hence writing $2x$ in place of x, we get

$$\frac{2x}{2} \cdot \frac{e^{2x}+1}{e^{2x}-1} = 1 + B_1\frac{(2x)^2}{2!} - B_3\frac{(2x)^4}{4!} + B_5\frac{(2x)^6}{6!} + \ldots$$

i.e. $$x\cot hx = 1 + B_1\frac{2^2x^2}{2!} - B_3\frac{2^4x^4}{4!} + B_5\frac{2^6x^6}{6!} - \ldots$$

Cor. 2. Expansion of $x\cot x$.

From cor. 1, we have

$$x\cot hx = 1 + B_1\frac{2^2x^2}{2!} - B_3\frac{2^4x^4}{4!} + B_5\frac{2^6x^6}{6!} - \ldots$$

Writing ix for x, we get

$$ix \cdot \frac{e^{2ix}+1}{e^{2ix}-1} = 1 + B_1\frac{2^2(ix)^2}{2!} - B_3\frac{2^4(ix)^4}{4!} + \ldots$$

$$\Rightarrow \quad ix \cdot \frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = 1 - B_1\frac{2^2x^2}{2!} - B_3\frac{2^4x^4}{4!} - \ldots$$

$\Rightarrow$ $x\cot x =$ " " " " "

Method III. Method of formation of Differential Equations

Under this method, we evaluate the nth differential coefficient of a given function by the use of Leibnitz Theorem and then putting $x = 0$ and putting the particular values of $n = 1, 2, 3, \ldots$ successively we determine $f(0)$, $f'(0)$, $f''(0)$ ... In the end, the expansion of the given function is found out by the use of Maclaurin's Theorem.

EXAMPLE 1. *Expand* $\tan^{-1}x$. **(... 71A; Mithila. 76A, 80A)**

SOLUTION : Let $y = f(x) = \tan^{-1}x$ $\quad\therefore\quad f(0) = 0$

Then $y_1 = \dfrac{1}{1+x^2}$ $\quad\therefore\quad f'(0) = 1.$

From this equation, $(1 + x^2)y_1 = 1$.

Differentiating this, $(1 + x^2)y_2 + y_1(2x) = 0$ $\quad\therefore\quad (y_2)_0 = 0.$

Now differentiating this n times, by the use of Leibnitz Theorem, we have

$$\left\{(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{1.2}\cdot 2y_n\right\} + 2\{xy_{n+1} + n(1)y_n\} = 0$$

$$\Rightarrow \quad (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0.$$

Putting $x = 0$ in this, $(y_{n+2})_0 = -n(n+1)(y_n)_0$ ...(1)

From the equation (1),

putting $n = 1$, $(y_3)_0 = -2(y_1)_0 = -2$

putting $n = 2$, $(y_4)_0 = -2(3)(y_2)_0 = 0$

putting $n = 3$, $(y_5)_0 = -3(4)(y_3)_0 = 2.3.4$

putting $n = 4$, $(y_6)_0 = 0$...etc.

Hence according to Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

i.e. $\tan^{-1}x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^5}{5!}2.3.4 + \ldots$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

EXAMPLE 2. *Expand* $\sin^{-1}x$.

(B.U. 67S, 69A; P.U. 73A; Mithila 77; R.U. 81; Bhag.U. 90)

SOLUTION : Let $y = f(x) = \sin^{-1}x$ $\therefore$ $f(0) = 0$.

Then $y_1 = \frac{1}{\sqrt{1-x^2}}$ $\therefore$ $f'(0) = 1$.

From the equation $\sqrt{1-x^2}.y_1 = 1 \Rightarrow (1-x^2)y_1^2 = 1$.

Differentiating this,

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 = 0$$

$\Rightarrow$ $(1-x^2)y_2 - xy_1 = 0$ $\therefore$ $(y_2)_0 = 0$.

Now differentiating this n times, according to Leibnitz Theorem, we have

$$\left\{(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n\right\} - \{xy_{n+1} + n(1)y_n\} = 0$$

$\Rightarrow$ $(1-x^2)y_{n+2} - xy_{n+1}(2n+1) - n^2y_n = 0$.

Putting $x = 0$ in this, $(y_{n+2})_0 = n^2(y_n)_0$ ...(1)

In equation (1),

putting $n = 1$, $(y_3)_0 = 1^2(y_1)_0 = 1$.

putting $n = 2$, $(y_4)_0 = 2^2(y_2)_0 = 0$

putting $n = 3$, $(y_5)_0 = 3^2(y_3)_0 = 3^2$

putting $n = 4$, $(y_6)_0 = 4^2(y_4)_0 = 0$

putting $n = 5$, $(y_7)_0 = 5^2(y_5)_0 = 3^2.5^2$... etc.

Hence according to Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

i.e. $\sin^{-1}x = 0 + x + 0 + \frac{x^3}{3}\cdot 1 + \frac{x^5}{5!}\cdot 3^2 + \frac{x^7}{7!}3^2.5^2 + \dots$

$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \dots$$

EXAMPLE 3. *Expand* $\sin h^{-1}x$ **(B.U. 70A)**

SOLUTION : Let $y = \sin h^{-1}x \quad \therefore\ f(0) = 0$

$\therefore\ y_1 = \dfrac{1}{\sqrt{1+x^2}} \qquad \therefore\ f'(0) = 1.$

$\Rightarrow\ \sqrt{1+x^2}\,y_1 = 1.$

Squaring, $(1+x^2)y_1^2 = 1.$

Again differentiating,

$$(1+x^2)2y_1y_2 + y_1^2(2x) = 0$$

$\Rightarrow\ (1+x^2)y_2 + xy_1 = 0 \quad \therefore\ (y_2)_0 = 0.$

Now differentiating this n times, according to Leibnitz Theorem, we have

$$\left\{(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{1.2}\cdot 2y_n\right\} + [xy_{n+1} + n(1)y_n] = 0$$

$\Rightarrow\ (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0.$

Putting $x = 0$ we get, $(y_{n+2})_0 = -n^2(y_n)_0$.

Now putting successively

$$n = 1,\ (y_3)_0 = -1^2.1 = -1$$

$$n = 3,\ (y_5)_0 = -3^2(y_3)_0 = +3^2 \text{ etc.}$$

Again putting $n = 2$, $(y_4)_0 = -2^2(y_2)_0 = 0$

putting $n = 4$, $(y_6)_0 = 0$ ... etc.

Hence according to Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$= 0 + x(1) + 0 + \frac{x^3}{3!}(-1) + 0 + \frac{x^5}{5!}\cdot 3^2 + \dots$$

$$= x - \frac{x^3}{3!} + 3^2\cdot\frac{x^5}{5!} - \dots$$

i.e. $\sin h^{-1}x = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \dots$

Cor. We know that $\sin h^{-1}x = \log(x + \sqrt{x^2+1})$.

$\therefore$ (i) $\log(x + \sqrt{x^2+1}) = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \ldots$

(ii) Putting $x = 1$ in this, we get

$$\log(1+\sqrt{2}) = 1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} - \ldots$$

EXAMPLE 4. *Expand* $(\sin^{-1}x)^2$ *in ascending powers of* x. **(P.U. 50H)**

SOLUTION : Let $y = (\sin^{-1}x)^2$.

Then $y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$ $\quad\therefore\quad \sqrt{1-x^2}\,.y_1 = 2\sin^{-1}x.$

Squaring, $(1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y.$

Again, differentiating

$$(1-x^2)2y_1y_2 + y_1^{\,2}(-2x) = 4y_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 - 2 = 0 \qquad \ldots(1)$$

Now differentiating (1) n times, according to Leibnitz Theorem, we have

$$\{(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n\} - \{xy_{n+1} + n(1)y_n\} = 0.$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

Putting $x = 0$ in this, $(y_{n+2})_0 = n^2(y_n)_0$ $\qquad \ldots(2)$

From the given equation, $f(0) = 0$, $(y_1)_0 = 0$, $(y_2)_0 = 2$.

Now putting $n = 1$ in (2), $(y_3)_0 = 1^2(y_1)_0 = 0$;

putting $n = 3$, $(y_5)_0 = 3^2(y_3)_0 = 0$ etc.

Again putting $n = 2$ in (2), $(y_4)_0 = 2^2(y_2)_0 = 2^2 2$;

putting $n = 4$, $(y_6)_0 = 4^2(y_4)_0 = 4^2.2^2.2$ etc.

Hence according to Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$= 0 + 0 + \frac{x^2}{2!}\cdot 2 + \frac{x^4}{4!}2^2 2 + \frac{x^6}{6!}4^2 2^2.2 + \ldots$$

$$= 2\frac{x^2}{2!} + \frac{2^2}{4!}2x^4 + \frac{2^2 4^2}{6!}\cdot 2x^6 + \ldots$$

$$= \frac{2x^2}{2} + \frac{2.2^2}{1.2.3}\cdot\frac{x^4}{4} + \frac{2.2.^2.4^2}{1.2.3.4.5}\cdot\frac{x^6}{6} + \ldots$$

$$= 2\left[\frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \ldots\right]$$

$$\therefore \quad \frac{(\sin^{-1}x)^2}{2} = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \ldots$$

Cor. Differentiating this expansion, we get

(i) $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \dfrac{2}{3}x^3 + \dfrac{2.4}{3.5}x^5 + \ldots$

(ii) Putting $x = \sin\theta$, we get

$$\frac{\theta}{\cos\theta} = \sin\theta + \frac{2}{3}\sin^3\theta + \frac{2.4}{3.5}\sin^5\theta + \ldots$$

EXAMPLE 5. *Expand* $\sin(m\sin^{-1}x)$ *in ascending powers of* x.

SOLUTION : Let $y = \sin(m\sin^{-1}x)$. ...(1)

Then $y_1 = \cos(m\sin^{-1}x) \times \dfrac{m}{\sqrt{1-x^2}}$ ...(2)

$\Rightarrow \quad y_1\sqrt{1-x^2} = m\cos(m\sin^{-1}x)$

$\Rightarrow \quad (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$= m^2\{1 - \sin^2(m\sin^{-1}x)\} = m^2(1-y^2).$

Again differentiating,

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2(-2yy_1)$$

$\Rightarrow \quad (1-x^2)y_2 - xy_1 + m^2y = 0.$ ...(3)

Now differentiating (3) n times, according to Leibnitz Theorem, we have,

$$\{(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n\}$$

$$- \{xy_{n+1} + n(1)y_n\} + m^2y_n = 0$$

$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Putting $x = 0$ in this, $(y_{n+2})_0 = -(m^2-n^2)(y_n)_0$ ...(4)

From (1), $f(0) = 0$.

From (2), when $x = 0$, $(y_1)_0 = m$.

From (3), when $x = 0$, $(y_2)_0 = 0$.

Now we shall put $n = 1, 2, 3, 4, \ldots$ successively in (4).

Putting $n = 1$ in (4), $(y_3)_0 = -(m^2-1^2)(y_1)_0 = m(m^2-1^2)$.

Putting $n = 3$, $(y_5)_0 = -(m^2-3^2)(y_3)_0$

$= +\, m(m^2-1^2)(m^2-3^2)$

.....

Again putting $n = 2$, $(y_4)_0 = (m^2 - 2^2)(y_2)_0 = 0$.

Similarly $(y_6)_0 = 0 = (y_8)_0$ etc.

Hence according to Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$= 0 + mx - m(m^2 - 1^2)\frac{x^3}{3!} + m(m^2 - 1^2)(m^2 - 3^2)\frac{x^5}{5!} + \ldots$$

i.e. $\sin(m\sin^{-1}x) =$ " " " " " " "

Cor. Putting $x = \sin\theta$ in this result, we get

$$\sin m\theta = m\sin\theta - \frac{m(m^2 - 1^2)}{3!}\sin^3\theta + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!}\sin^5\theta - \ldots$$

(Kanpur 83)

and this series is finite when m is an odd integer.

EXAMPLE 6. *If* $y = e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \ldots$; *then prove that*

(i) $(1 + x^2)y_2 + (2x - 1)y_1 = 0$.

(ii) $(1 + x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n + 1)y_n = 0$.

(iii) $(n + 2)a_{n+2} + na_n = a_{n+1}$ **(B.U. 53H; R.U. 72A)**

SOLUTION : Given that $y = e^{\tan^{-1}x}$.

$$\therefore \quad y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1 + x^2} = \frac{y}{1 + x^2}$$

$$\therefore \quad (1 + x^2)y_1 = y.$$

Again differentiating, we have

$$(1 + x^2)y_2 + 2x.y_1 = y_1$$

$$\therefore \quad (1 + x^2)y_2 + (2x - 1)y_1 = 0 \qquad \ldots(1)$$

Thus part (i) is proved.

Now differentiating the equation (1) n times, according to Leibnitz Theorem, we have

$$(1 + x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{1.2} \cdot 2y_n + (2x - 1)y_{n+1} + n(2)y_n = 0$$

$$\Rightarrow \quad (1 + x^2)y_{n+2} + y_{n+1}(2nx + 2x - 1) + y_n\{n(n - 1) + 2n\} = 0$$

$$\Rightarrow \quad (1 + x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n + 1)y_n = 0. \qquad \ldots(2)$$

This proves the part (ii).

Now putting $x = 0$ in (2),

$$(y_{n+2})_0 - (y_{n+1})_0 + n(n+1)(y_n)_0 = 0. \qquad ...(3)$$

But it is given that $y = a_0 + a_1x + a_2x^2 + \ldots$

$\therefore \quad (y_n)_0 = a_n\, n!,\ (y_{n+1})_0 = a_{n+1}(n+1)!\ .$

$$(y_{n+2})_0 = a_{n+2}(n+2)!\ .$$

Hence from (3),

$$a_{n+2}(n+2)! - a_{n+1}(n+1)! + n(n+1)a_n n! = 0$$

$\therefore \quad (n+2)(n+1)a_{n+2} - a_{n+1}(n+1) + n(n+1)a_n = 0$

$\Rightarrow \quad (n+2)a_{n+2} - a_{n+1} + na_n = 0$

$\Rightarrow \quad (n+2)a_{n+2} + na_n = a_{n+1}.$

This proves the part (iii).

EXAMPLE 7. *Expand* $e^{a\sin^{-1}x}$.

(B.U. 61S; P.U. 45S; M.U. 67A, 77A; R.U. 75A; M.U. 77)

SOLUTION : Let $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad ...(1)$

Then $y_1 = e^{a\sin^{-1}x} \times a\ \frac{1}{\sqrt{1-x^2}}$

$$= \frac{ay}{\sqrt{1-x^2}} \quad \{= a_1 + a_2\, 2x + \ldots\}$$

$\Rightarrow \quad \sqrt{1-x^2}\ y_1 = ay \Rightarrow (1-x^2)y_1^{\,2} = a^2y^2.$

Again differentiating,

$$(1-x^2).2y_1y_2 + y_1^{\,2}(-2x) = a^2 2yy_1$$

$\Rightarrow \quad (1-x^2)y_2 - xy_1 - a^2y = 0.$

Again, differentiating n times, according to Leibnitz Theorem, we have

$$\{(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n\}$$

$$- \{xy_{n+1} + n(1)y_n\} - a^2y_n = 0$$

$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2y_n = 0$

$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$

Putting $x = 0$ in this, we have

$$(y_{n+2})_0 = (n^2 + a^2)(y_n)_0. \qquad ...(2)$$

But from (1), since $y = a_0 + a_1x + a_2x^2 + \ldots$

$\therefore \quad y_{n+2} = a_{n+2}(n+2)! + a_{n+3}(n+3)!x + \ldots$

and $\quad y_n = a_n n! + a_{n+1}(n+1)!x + \ldots$

Therefore $(y_{n+2})_0 = a_{n+2}(n+2)!$ and $(y_n)_0 = a_n n!$.

Hence from (2),

$$a_{n+2}(n+2)! = (n^2 + a^2).a_n n!$$

$$\Rightarrow \quad (n+1)(n+2)a_{n+2} = (n^2 + a^2)a_n. \qquad \textbf{(P.U. 60A, 64S)}$$

$$\therefore \quad a_{n+2} = \frac{a^2 + n^2}{(n+1)(n+2)}a_n \qquad \ldots(3)$$

Putting $x = 0$ in (1), $a_0 = 1$

and $(y_1)_0 = a.\ 1 = a$ i.e. $a_1 = a$.

Now putting successively $n = 0, 1, 2, 3, \ldots\ldots$ in (3), we get

putting $n = 0$, $a_2 = \frac{a^2}{2!}$;

putting $n = 1$, $a_3 = \frac{a^2 + 1^2}{2.3}a_1 = \frac{a(a^2 + 1^2)}{3!}$;

putting $n = 2$, $a_4 = \frac{a^2 + 2^2}{3.4}a_2 = \frac{a^2(a^2 + 2^2)}{4!}$;

putting $n = 3$, $a_5 = \frac{a^2 + 3^2}{4.5}a_3 = \frac{a(a^2 + 1^2)(a^2 + 3^2)}{5!}$ etc.

Hence putting the values of $a_0, a_1, a_2, a_3, \ldots\ldots$ in (1),

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(a^2 + 1^2)}{3!}x^3$$

$$+ \frac{a^2(a^2 + 2^2)}{4!}x^4 + \frac{a(a^2 + 1^2)(a^2 + 3^2)}{5!}x^5 + \ldots\ldots$$

Cor. 1. Putting $a = 1$,

$$e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \frac{2}{3!}x^3 + \frac{5}{4!}x^4 + \frac{20}{5!}x^5 + \ldots\ldots \qquad \textbf{(B. U. 61S)}$$

Cor. 2. Putting $\sin^{-1}x = \theta$ above,

$$e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + \frac{20}{5!}\sin^5\theta + \ldots\ldots \qquad \textbf{(Kanpur 87)}$$

EXAMPLE 8. *If* $e^{e^x} = \Sigma a_n x_n$, *prove that*

$$a_{n+1} = \frac{1}{n+1}\left[a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \frac{a_{n-3}}{3!} + \ldots + \frac{a_0}{n!}\right]$$

(B.U. 60H, 64H, 66H; R.U. 62H; M.U. 69H, Bhag. 72H)

SOLUTION : Given

$$e^{e^x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1} + a_nx^n + a_{n+1}x^{n+1} + \ldots$$

Differentiating both sides w.r.t. x, we get

$$e^{e^x} \times e^x = a_1 + 2a_2x + 3a_3x^2 + \ldots + (n-1)a_{n-1}x^{n-2}$$

$$+ na_nx^{n-1} + (n+1)a_{n+1}x^n + \ldots$$

$$\Rightarrow (a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + a_{n+1}x^{n+1} + \ldots) \times$$

$$\left\{1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \ldots\right\}$$

$$= a_1 + 2a_2x + 3a_3x^2 + \ldots + (n-1)a_{n-1}x^{n-2}$$

$$+ na_nx^{n-1} + (n+1)a_{n+1}x^n + \ldots$$

We shall now equate the coeff. of x^n from both sides.

The coeff. of x^n on R.H.S. $= (n+1)a_{n+1}$ ...(1)

The coeff. of x^n on L.H.S. $= \frac{a_0}{n!} + \frac{a_1}{(n-1)!} + \frac{a_2}{(n-2)!} + \ldots + \frac{a_{n-1}}{1!} + a_n$ ...(2)

Hence equating the coeff. of x^n from both sides as given in (1) and (2), we get the result.

Method IV. Differentiation or Integration of a known Series

We shall assume here that we can differentiate or integrate term-by-term an infinite series.

EXAMPLE 1. *Expand* $\sin^{-1}x$. **(P.U. 73A; B.U. 67A, 69S; Bhag. 69S; Mithila. 77)**

SOLUTION : Let $f(x) = \sin^{-1}x$.

Then $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$$= 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}+1)}{1.2}x^4 + \ldots \text{ where } |x^2| < 1.$$

$$= 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \ldots$$

Now integrating term-by-term this series, we get

$$f(x) = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \ldots$$

Hence $\sin^{-1}x =$ " " " "

Note : The constant of integeration will be zero when $x = 0$.

EXAMPLE 2. *Expand* $\tan^{-1}x$ *in ascending powers of* x.

(P.U. 48A; Bhag. 71A; Mithila. 76A, 80A)

SOLUTION : Let $f(x) = \tan^{-1}x$.

Then $f'(x) = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$

$= 1 - x^2 + x^4 - x^6 + \ldots$. where $-1 < x < 1$.

Again integrating the series term-by-term

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Hence $\tan^{-1}x =$ " " " "

Note : The constant of integration will be zero when $x = 0$.

Example 6

1. Use Taylor's Theorem to expand e^{x+h} in powers of h.

(B.U. 57S, 66A; R.U. 66S; P.U. 80A)

2. Prove the following results by Taylor's Theorem

(i) $\sin(x+h) = \sin x + h\cos x - \dfrac{h^2}{2!}\sin x + \ldots$ **(Bhag. 91; Gorakhpur 87)**

(ii) $\sin^{-1}(x+h) = \sin^{-1}x + \dfrac{h}{\sqrt{1-x^2}} + \dfrac{x}{(1-x^2)^{3/2}} \cdot \dfrac{h^2}{2!}$

$+ \dfrac{1+2x^2}{(1-x^2)^{5/2}} \cdot \dfrac{h^3}{3!} + \ldots$ **(R.U. 71S; Bhag. 88, M.U. 90)**

(iii) $\sec^{-1}(x+h) = \sec^{-1}x + \dfrac{h}{x\sqrt{x^2-1}} - \dfrac{2x^2-1}{x^2(x^2-1)^{3/2}} \cdot \dfrac{h^2}{2!} + \ldots$

Prove that

3. $\log(1 - x + x^2) = -x + \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \dfrac{x^4}{4} - \dfrac{x^5}{5} \ldots$

4. $\log\dfrac{\sin x}{x} = -\dfrac{x^2}{6} - \dfrac{x^4}{180} - \ldots$

5. $\log\dfrac{\tan x}{x} = \dfrac{x^2}{3} + \dfrac{7}{90}x^4 + \ldots$ **(P.U. 53)**

6. $\log\dfrac{\tan^{-1}x}{x} = -\dfrac{x^2}{3} + \dfrac{13}{90}x^4 - \ldots$

7. $\log\dfrac{\sin hx}{x} = \dfrac{x^2}{6} - \dfrac{x^4}{180} - \ldots$

8. $\log\sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \ldots$

9. $\tan h^{-1}x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$

10. $\dfrac{1}{2}\{\log(1+x)\}^2 = \dfrac{x^2}{2} - \left(1+\dfrac{1}{2}\right)\dfrac{x^3}{3} + \left(1+\dfrac{1}{2}+\dfrac{1}{3}\right)\dfrac{x^4}{4} - \ldots$

11. $e^x \sin x = x + \dfrac{2x^2}{2!} + \dfrac{2x^3}{3!} - \dfrac{4x^5}{5!} - \ldots$

(R.U. 62A; Bhag. 63S; Gorakhpur 81)

12. $e^x \log(1+x) = x + \dfrac{x^2}{2!} + \dfrac{2x^3}{3!} + \ldots$ **(P.U. 59S)**

13. $\cos x \cos hx = 1 - \dfrac{2^2x^4}{4!} + \dfrac{2^4x^8}{8!} - \dfrac{2^6x^{12}}{12!} + \ldots$

[**Hint :** Use $\cos hx = \dfrac{e^x + e^{-x}}{2}$ and then write the n th diff. coeff. of $e^x \cos x$ etc.]

14. $\sin x \cos hx = x + \dfrac{2x^3}{3!} - \dfrac{4x^5}{5!} + \ldots$ **(P.U. 57A; B.U. 55S)**

15. $e^{ax}\cos hx = 1 + ax + (a^2+1)\dfrac{x^2}{2!} + (a^3+3a)\dfrac{x^3}{3!} + \ldots$

(Bhag. 63A; B.U. 54A)

16. $e^{ax}\sin bx = bx + abx^2 + \dfrac{3a^2b - b^3}{3!}x^3 + \ldots$

$$+ \frac{(a^2+b^2)^{n/2}}{n!}x^n \sin\left(n\tan^{-1}\frac{b}{a}\right) + \ldots$$

17. $e^{\sec x} = e\left\{1 + \dfrac{x^2}{2!} + \dfrac{8x^4}{4!} + \ldots\right\}$ **(B.U. 63S)**

18. (i) $e^{x\sin x} = 1 + x^2 + \dfrac{1}{3}x^4 + \dfrac{1}{120}x^6 + \ldots$

(ii) $e^{x\sec x} = 1 + x + \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3 + \ldots$ **(B.U. 65S, 74A)**

19. $\tan\left(\dfrac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \dfrac{8}{3}x^3 + \dfrac{10}{3}x^4 + \ldots$ **(P.U. 55A)**

20. $\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots$

21. Expand $e^{ax}\cos bx$ by Maclaurin's Theorem and deduce that

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots$$

22. Prove that

(i) $x\cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 + \ldots$ **(R.U. 70; M.U. 75A)**

(ii) $x\operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \ldots$ **(P.U. 57)**

(iii) Expand $x\cot bx$ in a series in ascending powers of x up to the fourth power. **(P.U. 58A)**

23. Expand in powers of x

(i) $\sin^{-1}\frac{2x}{1+x^2}$ (ii) $\cos^{-1}\frac{1-x^2}{1+x^2}$ (iii) $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

24. Expand $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ in ascending integral powers of x.

(P.U. 54S; Bhag. 62S)

Hint : Let $x = \sin\theta$, so that

$$f(x) = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}\frac{\sin\theta}{\cos\theta}$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}x]$$

25. Expand in ascending integral powers of x.

(i) $\tan^{-1}x + \tanh^{-1}x$ (ii) $\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$

(iii) $\tan^{-1}\frac{2x}{1-x^2} + \cos^{-1}\frac{1-x^2}{1+x^2}$ (iv) $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

(v) $\sin^{-1}(3x - 4x^3)$.

26. Prove that

(i) $\frac{(\sinh^{-1}x)^2}{2!} = \frac{x^2}{2} - \frac{2}{3}\cdot\frac{x^4}{4!} + \frac{2.4}{3.5}\cdot\frac{x^6}{6!} - \ldots.$

(ii) $\frac{\sinh^{-1}x}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 - \ldots$

27. If $y = \cos(m\sin^{-1}x)$, prove that

(i) $(1 - x^2)y_2 - xy_1 + m^2y = 0$ **(P. U. 56A)**

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$.

Hence expand $\cos(m\sin^{-1}x)$ in ascending powers of x.

Deduce the expansion of $\cos m\theta$.

28. (i) Expand $\log\{x + \sqrt{1 + x^2}\}$ by Maclaurin's Theorem in ascending powers of x and find the general term. **(B.U. 72A)**

(ii) Expand $\dfrac{\log(1 + x)}{1 + x}$ in powers of x as far as x^4.

(Bhag. 66A; P.U. 50H; B.U. 91H)

29. Expand $\log[1 - \log(1 - x)]$ in powers of x by Maclaurin's Theorem up to the term of x^3 and deduce the expansion of $\log[1 + \log(1 + x)]$.

[**Hint :** Let $y = \log[1 - (1 - x)]$.

Then differentiating w.r.t.x,

$$y_1 = \frac{1}{1 - \log(1 - x)} \times \frac{1}{1 - x} = \frac{1}{e^y} \cdot \frac{1}{1 - x}$$

$\therefore\ (1 - x)y_1 = e^{-y}$ etc.

For the second part we substitute $\dfrac{x}{1 + x}$ for x].

30. If $y = \sin\log(x^2 + 2x + 1)$, prove that

$$(x + 1)^2y_{n+2} + (2n + 1)(x + 1)y_{n+1} + (n^2 + 4)y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as far as x^6.

[From the given equation, $y = \sin[2\log(x + 1)]$.

Then $y_1 = \cos[2\log(x + 1)] \cdot \dfrac{2}{x + 1}$.

$\therefore\ (x + 1)y_1 = 2\cos[2\log(x + 1)]$

$\therefore\ (x + 1)^2y_1^2 = 4\cos^2[2\log(x + 1)]$

$= 4[1 - \sin^2\{2\log(x + 1)\}]$

$= 4(1 - y^2)$].

50. Expand $\sin x$ in powers of $x - \frac{\pi}{2}$.

[Let $f(x) = \sin x = \sin\left\{\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right\}$, putting in the form $\sin(x + h)$.

Then from Taylor's Theorem

$$\sin(x + h) = \sin x + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

Hence $$\sin\left\{\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right\} = \sin\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \ldots$$

51. Expand e^x in ascending powers of $x - 1$. **(M.U.73A)**

52. Prove that $f(mx) = f(x) + (m - 1)f'(x) + \frac{(m-1)^2}{2!}x^2 f''(x) + \ldots\ldots$

(Kanpur 88)

53. If $e^{\sin x} = \Sigma a_n x^n$, prove that

$$a_n - \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} - \frac{a_{n-3}}{3!} + \ldots = \frac{\sin\frac{n\pi}{2}}{n!}.$$

(R.U. 85; M. U. 63H, 69H, 73H)

54. If $x^2 + 2x = 2 \log\left(c\frac{dx}{dy}\right)$ and $y = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \ldots$,

show that $a_{n+2} + a_{n+1} + na_n = 0$. **(R.U. 64H)**

[Hints: From the given equation, we have

$$x^2 + 2x = 2\{\log c - \log y_1\} = 2 \log c - 2 \log y_1$$

Differentiating, $2x + 2 = -2\frac{y_2}{y_1} \Rightarrow y_2 + (x + 1)y_1 = 0.$

Now, differentiating n times further,

$$y_{n+2} + (x + 1)y_{n+1} + {}^nc_1 \cdot 1 \cdot y_n = 0$$

$$\Rightarrow (y_{n+2})_0 + (y_{n+1})_0 + n(y_n)_0 = 0.]$$

ANSWER

1. $e^a\left[1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots\right]$

21. $e^{ax}\cos bx = 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + (a^3 - 3ab^2)\frac{x^3}{3!} + \ldots$

22. (iii) $1 + \frac{1}{6}\cdot\frac{2^2x^2}{2!} - \frac{1}{30}\cdot\frac{2^4x^4}{4!} + \ldots$

23. (i) $2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$ (ii) $2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$

(iii) $\frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$

24. $x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \ldots$

25. (i) $2\left(x + \frac{x^5}{5} + \frac{x^9}{9} + \ldots\right)$ (ii) $3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$

(iii) $4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$ (iv) The given exp. $= \frac{1}{2}\sin^{-1}(x^2)$

(v) The given exp. $= 3\sin^{-1}x$.

27. $\cos(m\sin^{-1}x) = 1 - \frac{m^2x^2}{2!} + \frac{m^2(m^2 - 2^2)}{4!}x^4$

$$- \frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{6!}x^6 +$$

28. (i) $x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \ldots$ (ii) $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{50}{24}x^4 + \ldots$

29. $\log[1 - \log(1 - x)] = x + \frac{x^3}{6} + \ldots$

$\log[1 + \log(1 + x)] = x - x^2 + \frac{7x^3}{6} - \ldots$

32. $e\left[1 + (x - 1) + \frac{(x - 1)^2}{2!} + \frac{(x - 1)^3}{3!} + \ldots\right]$

3

PARTIAL DIFFERENTIATION

3.1. So far we had been dealing with the differentiation of functions of one variable only. But on numerous occassions, we come across functions of two variables or more than two variables. To take a simple example from a physical world, the price of an object depends upon the demand and supply of that object. Hence the price of an object is a function of two variables namely the demand and supply of that object. Similar examples of a function can be cited where three or more variables are involved.

We know how to evaluate $\frac{dy}{dx}$ when $y = f(x)$ is a function of a single variable x. But if $u = f(x, y)$ be a function of two variables x and y or if $u = f(x, y, z)$ be a function of three variables x, y, z etc., then the problem is how to evaluate the total differential du in these cases. Before this, it becomes necessary to define the partial derivative of u w.r. to x or w.r. to y or w.r. to z. Therefore at the outset, we shall define the partial derivative of u.

Definition : Let $u = f(x, y)$ be a function of two variables x and y.

The partial derivative of u w.r. to x is denoted by the symbol $\frac{\partial u}{\partial x}$ and this means that u has been differentiated w.r. to x only and the remaining variable y has been treated as constant.

That is, if $u = f(x, y)$, then

$$\frac{\partial u}{\partial x} = \underset{h \to 0}{Lt} \frac{f(x + h, y) - f(x, y)}{h}$$

if this limit exists.

Similarly, the partial derivative of u w.r. to y is denoted by $\frac{\partial u}{\partial y}$ and this means that u has been differentiated w.r. to y only and the remaining variable x has been treated as constant.

That is, if $u = f(x, y)$, then

$$\frac{\partial u}{\partial y} = \underset{k \to 0}{Lt} \frac{f(x, y+k) - f(x, y)}{k}$$

if this limit exists.

Now, let $u = f(x, y, z)$ be any function of three variables x, y, z. Then, $\frac{\partial u}{\partial x} = u$ has been differentiated w.r. to x only while the remaining variables y and z have been treated as constants.

That is,

$$\frac{\partial u}{\partial x} = \underset{h \to 0}{Lt} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similarly

$\frac{\partial u}{\partial y} = u$ has been differentiated w.r. to y only while the remaining variables x and z have been treated as constants.

That is,

$$\frac{\partial u}{\partial y} = \underset{k \to 0}{Lt} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

and $\frac{\partial u}{\partial z} = u$ has been differentiated w.r to z only while the remaining variables x and y have been treated as constants.

That is,

$$\frac{\partial u}{\partial z} = \underset{l \to 0}{Lt} \frac{f(x, y, z+l) - f(x, y, z)}{l}.$$

In the same manner, if u is a function of more than three variables, then the definition of partial derivative for the function u can be likewise extended. Generally, if

$$u = f(x_1, x_2, x_3, \ldots x_n)$$

is a function of n variables $x_1, x_2, x_3, \ldots x_n$, then $\frac{\partial u}{\partial x_1}$ means that u has been differentiated w.r. to x_1 only while the remaining variables $x_2, x_3, \ldots x_n$ have been treated as constants.

Similarly $\frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots \frac{\partial u}{\partial x_n}$ can be defined.

EXAMPLE 1. *Find* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ *when* $u = x^3 + y^3 + 3axy$.

SOLUTION : Given that $u = x^3 + y^3 + 3axy$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 + 0 + 3ay.1 \qquad (\because\ y \text{ has beeen treated as constant})$$

$$= 3x^2 + 3ay = 3(x^2 + ay)$$

and $$\frac{\partial u}{\partial y} = 0 + 3y^2 + 3ax.1 \qquad (\because\ x \text{ has been treated as constant})$$

$$= 3y^2 + 3ax = 3(y^2 + ax).$$

EXAMPLE 2. *Find* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ *when* $u = \log(x^3 + y^3 + z^3 - 3xyz)$..

SOLUTION : According to the rule of differentiating a function of function, we have

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \frac{3x^2 + 0 + 0 - 3yz}{x^3 + y^3 + z^3 - 3xyz};$$

(since y and z have been treated as constants)

$$= \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}.$$

Similarly $$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times \frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \frac{0 + 3y^2 + 0 - 3xz}{x^3 + y^3 + z^3 - 3xyz}.$$

(Here x and z have been treated as constants)

$$= \frac{3(y^2 - xz)}{x^3 + y^3 + z^3 - 3xyz}.$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times \frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \frac{0 + 0 + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

(Hence x and y have been treated as constants)

$$= \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}.$$

Note : $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$.

$$=\frac{3(x^2-yz+y^2-zx+z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$=\frac{3}{x+y+z}.$$

3.2. Partial Differentiation of higher orders

Here we shall confine ourselves to the partial differentiation of second order.

We know that $\dfrac{\partial u}{\partial x}$ means that u has been differentiated w.r. to x only while y is constant..

Now, if we differentiate $\dfrac{\partial u}{\partial x}$ partially w.r. to x (partial differentiation w.r to x will mean that y has been treated as constant again), then that is denoted by $\dfrac{\partial^2 u}{\partial x^2}$. Thus

$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$ which means that $\dfrac{\partial u}{\partial x}$ has been differentiated w.r.t x only while y has been treated as constant.

Similarly,

$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$ which means that $\dfrac{\partial u}{\partial y}$ has been differentiated w.r. to y only while x has been treated as constant.

$\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ which means that $\dfrac{\partial u}{\partial y}$ has been differentiated partially w.r. to x, while differentiating $\dfrac{\partial u}{\partial y}$ partially w.r to x, y will be treated as constant.

$\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$; this is different from the previous one, for here first of all $\dfrac{\partial u}{\partial x}$ will be calculated; then $\dfrac{\partial u}{\partial x}$ will be differentiated partially w.r to y. Thus while differentiating $\dfrac{\partial u}{\partial x}$ partially w.r. to y, x will be treated as constant.

Note : This is not necessary that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

However, for our purposes, we shall assume that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

EXAMPLE *Calculate* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial y^2}$ *when* $u = x^3 + y^3 - 3axy$.

SOLUTION : From the definition,

$$\frac{\partial u}{\partial x} = 3x^2 + 0 - 3ay = 3(x^2 - ay)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 3(2x - 0) = 6x.$$

Again $$\frac{\partial u}{\partial y} = 0 + 3y^2 - 3ax = 3(y^2 - ax)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = 3(2y - 0) = 6y.$$

Worked Out Examples

EXAMPLE 1. *If* $u = \log \frac{x^2 + y^2}{xy}$, *verify that* $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$. **(Kanpur 88)**

SOLUTION : Here $u = \log(x^2 + y^2) - \log xy$ ...(1)

Differentiating (1) partially w.r. to x, (treating y as const.)

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{xy} \cdot y$$

$$= \frac{2x}{x^2 + y^2} - \frac{1}{x}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}\left\{\frac{2x}{x^2 + y^2} - \frac{1}{x}\right\}$$

$$= 2x\left\{-\frac{1}{(x^2 + y^2)^2} \cdot 2y\right\} \text{ ; treating } x \text{ as const.}$$

$$= -\frac{4xy}{(x^2 + y^2)^2} \qquad \ldots(2)$$

Similarly from (1),

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} \cdot 2y - \frac{1}{xy} \cdot x = \frac{2y}{x^2 + y^2} - \frac{1}{y}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left\{\frac{2y}{x^2+y^2} - \frac{1}{y}\right\}$$

$$= 2y\left\{-\frac{1}{(x^2+y^2)^2}\cdot 2x\right\} \text{ treating } y \text{ as const.}$$

$$= -\frac{4xy}{(x^2+y^2)^2} \qquad \ldots(3)$$

Now, from (2) and (3),

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

EXAMPLE 2. *If* $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, *prove that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$. **(Bhag. 64A)**

SOLUTION : From the given relation,

$$\frac{\partial u}{\partial y} = \left\{x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} + \tan^{-1}\frac{y}{x} \times 0\right\}$$

$$- \left\{y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} x\left(-\frac{1}{y^2}\right) + \tan^{-1}\frac{x}{y} \times 2y\right\}; \text{treating } x \text{ as const.}$$

$$= \frac{x^4}{x^2+y^2}\cdot\frac{1}{x} - \frac{y^4}{x^2+y^2}\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\frac{x}{y}.$$

Hence $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left\{x - 2y\tan^{-1}\frac{x}{y}\right\}$

$$= 1 - 2y\left\{\frac{1}{1+x^2/y^2}\cdot\frac{1}{y}\right\}; \text{ treating } y \text{ as const.}$$

$$= 1 - 2y\cdot\frac{y^2}{x^2+y^2}\cdot\frac{1}{y} = 1 - 2\frac{y^2}{x^2+y^2}$$

$$= \frac{x^2 - y^2}{x^2+y^2}.$$

EXAMPLE 3. *If* $u = e^{xyz}$, *show that*

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

(P.U. 63A; R.U. 69S, 72H, 84; Gorakhpur 81)

SOLUTION : Given that $u = e^{xyz}$ ...(1)

Differentiating (1) partially w.r.to z (treating x and y as constant).

$$\frac{\partial u}{\partial z} = e^{xyz}.xy. \qquad ...(2)$$

Now, differentiating (2) partially w.r to y (treating x and z as constant)

$$\frac{\partial^2 u}{\partial y \partial z} = x\{e^{xyz}.1 + ye^{xyz}.xz\}$$

$$= xe^{xyz} + x^2yze^{xyz} \qquad ...(3)$$

Again differentiating (3) partially w.r.to x (treating y and z as const.)

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \{xe^{xyz}.yz + e^{xyz}.1\} + yz(x^2e^{xyz}.yz + e^{xyz}.2x)$$

$$= e^{xyz}\{(xyz + 1) + (x^2y^2z^2 + 2xyz)\}$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2).$$

EXAMPLE 4. *If* $u = f\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial u} = 0$.

SOLUTION : Given that $u = f\left(\frac{y}{x}\right)$ ...(1)

Therefore differentiating partially w.r.to x (treating y as const.)

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right) \cdot y \cdot \left(-\frac{1}{x^2}\right) = -\frac{y}{x^2}f'\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{y}{x}f'\left(\frac{y}{x}\right) \qquad ...(2)$$

Again differentiating (1) partially w.r.to y, (treating x as const.)

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = \frac{1}{x}f'\left(\frac{y}{x}\right)$$

$$\therefore \quad y\frac{\partial u}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right) \qquad ...(3)$$

Now, adding (2) and (3),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

EXAMPLE 5. *If* $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

(Bhag. 65S, 66, 71; M.U. 67A, 71A; B.U. 63, 69S; R.U. 69A, 72A; Gorakhpur 81)

SOLUTION : Given that $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ ...(1)

Therefore differentiating partially w.r.to x,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{[1-(x/y)^2]}} \cdot \frac{1}{y} + \frac{1}{1+(y/x)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y} - \frac{x^2}{x^2 + y^2} \cdot \frac{y}{x^2}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}.$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \qquad ...(2)$$

Again differentiating (1) partially w.r.to y, we have

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{[1-(x/y)^2]}} \cdot x\left(-\frac{1}{y^2}\right) + \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= -\frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{x}{y^2} + \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}.$$

$$\therefore \quad y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \qquad ...(3)$$

Now adding (2) and (3).

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

EXAMPLE 6. *If* $z(x + y) = x^2 + y^2$, *show that*

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$ **(R.U. 715)**

SOLUTION : From the given relation, $z = \frac{x^2 + y^2}{x + y}$.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2 + y^2)(1)}{(x+y)^2}; \text{ treating } y \text{ as const.}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

Similarly $\frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2}$; treating x as const.

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$\therefore \quad \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{(x^2 + 2xy - y^2) - (y^2 + 2xy - x^2)}{(x+y)^2}$$

$$= \frac{2(x^2 - y^2)}{(x+y)^2} = \frac{2(x-y)}{x+y}.$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}. \qquad \ldots(1)$$

Again $1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}. \qquad \ldots(2)$$

Hence form (1) and (2),

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

EXAMPLE 7. *If* $x = r\cos\theta$, $y = r\sin\theta$, *show that*

(i) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$ *and* (ii) $\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$. **(M.U. 77H)**

SOLUTION : Given $x = r\cos\theta$. Therefore differentiating partially w.r.t. r,

$$\frac{\partial x}{\partial r} = \cos\theta. \qquad \ldots(1)$$

Again, from the given equations, we have $x^2 + y^2 = r^2$.

Therefore differentiating partially w.r.t. x, we get

$$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta. \qquad \ldots(2)$$

$\therefore$ From (1) and (2), $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$.

Thus (i) is proved.

Again from the equation $x = r\cos\theta$, we have

$$\frac{\partial x}{\partial \theta} = -\, r\sin\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta. \qquad \ldots(3)$$

Also, from the given equations, we have $\tan\theta = \frac{y}{x}$.

Differentiating partially w.r.t. x, we get

$$\sec^2\theta\frac{\partial \theta}{\partial x} = y\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2\sec^2\theta} = -\frac{y}{x^2(1+\tan^2\theta)}$$

$$= -\frac{y}{x^2(1+y^2/x^2)} = -\frac{y}{x^2+y^2}$$

$$= -\frac{y}{r^2} = -\frac{1}{r}\sin\theta$$

$$\Rightarrow \quad r\frac{\partial \theta}{\partial x} = -\sin\theta. \qquad \ldots(4)$$

From (3) and (4), (ii) is proved.

EXAMPLE 8. *If* $u = \phi(y + ax) + \psi(y - ax)$, *show that* $\dfrac{\partial^2 u}{\partial x^2} - a^2\dfrac{\partial^2 u}{\partial y^2} = 0$. **(B.U. 73)**

SOLUTION : Given that $u = \phi(y + ax) + \psi(y - ax)$. ...(1)

Differentiating (1) partially w.r.to x;

$$\frac{\partial u}{\partial x} = a\phi'(y + ax) - a\psi'(y - ax)$$

and $$\frac{\partial^2 u}{\partial x^2} = a^2\phi''(y + ax) + a^2\psi''(y - ax)$$

$$= a^2[\phi''(y + ax) + \psi''(y - ax)]. \qquad \ldots(2)$$

Again differentiating (1) partially w.r.to y,

$$\frac{\partial u}{\partial y} = \phi'(y + ax) + \psi'(y - ax)$$

and $$\frac{\partial^2 u}{\partial y^2} = \phi''(y + ax) + \psi''(y - ax) \qquad \ldots(3)$$

$\therefore$ From (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}.$$

Hence the result.

EXAMPLE 9. *If $u = f(r)$ where $r^2 = x^2 + y^2$, show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r). \qquad \textbf{(Kanpur 85)}$$

SOLUTION : Given that $r^2 = x^2 + y^2$.

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Now, since $u = f(r)$, therefore

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r} f'(r).$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left\{\frac{xf'(r)}{r}\right\}$$

$$= \frac{\dfrac{\partial}{\partial x}\{xf'(r)\}\, r - xf'(r)\dfrac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r\left[xf''(r)\dfrac{\partial r}{\partial x} + f'(r)\right] - xf'(r)\dfrac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r\left[xf''(r)\dfrac{x}{r} + f'(r)\right] - xf'(r)\dfrac{x}{r}}{r^2}$$

$$= \frac{1}{r^2}\left[x^2 f''(r) + rf'(r) - \frac{x^2}{r} f'(r)\right] \text{ since } \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{r^2}\left[y^2 f''(r) + rf'(r) - \dfrac{y^2}{r} f'(r)\right]$;

Adding, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

$$= \frac{1}{r^2}\left[(x^2 + y^2) f''(r) + 2rf'(r) - \frac{x^2 + y^2}{r} f'(r)\right]$$

$$= \frac{1}{r^2}\left[r^2 f''(r) + 2rf'(r) - \frac{r^2}{r} f'(r)\right]$$

$$= f''(r) + \frac{1}{r} f'(r) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

EXAMPLE 10. *If* $r^2 = x^2 + y^2 + z^2$, *show that*

$$\text{(i)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} = 0$$

$$\text{(ii)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r = \frac{2}{r}.$$ **(R.U. 67; M.U. 72H)**

SOLUTION : Here $r^2 = x^2 + y^2 + z^2$.

Differentiating partially w.r.t. x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly $$\frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Therefore $$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left\{\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) = -\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$= -\left\{\frac{1}{r^3} + x\frac{-3}{r^4}\frac{\partial r}{\partial x}\right\} = -\frac{1}{r^3} + \frac{3x}{r^4}\cdot\frac{x}{r}$$

$$= -\frac{1}{r^3} + \frac{3x^2}{r^5} \quad ...(1)$$

Similarly $$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3y^2}{r^5} \quad ...(2)$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + 3\frac{z^2}{r^5} \quad ...(3)$$

Adding (1), (2) and (3), we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} = -\frac{3}{r^3} + \frac{3}{r^5}(x^2 + y^2 + z^2)$$

$$= -\frac{3}{r^3} + \frac{3}{r^5}\cdot r^2 = -\frac{3}{r^3} + \frac{3}{r^3} = 0.$$

which proves (i).

For the second part, we see that $\frac{\partial r}{\partial x} = \frac{x}{r}$.

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \frac{r\cdot 1 - x\cdot\dfrac{\partial r}{\partial x}}{r^2} = \frac{1}{r} - \frac{x}{r^2}\cdot\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} \quad ...(4)$$

Similarly $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r} - \dfrac{y^2}{r^3}$ ...(5)

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{z^2}{r^3} \qquad \text{...(6)}$$

Adding (5), (6) and (7), we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r = \frac{3}{r} - \frac{1}{r^3}(x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{1}{r^3} \cdot r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

EXAMPLE 11. *If* $\dfrac{1}{u} = \sqrt{x^2 + y^2 + z^2}$, *show that*

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} + u = 0.$ **(M.U. 66)**

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$ **(B.U. 62S; P.U. 60A; M.U. 81; R.U. 85)**

SOLUTION : Given $\dfrac{1}{u} = (x^2 + y^2 + z^2)^{1/2}$.

$\therefore \quad \dfrac{1}{u^2} = x^2 + y^2 + z^2$

Hence differentiating partially w.r.to x, we get

$$-\frac{2}{u^3}\frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial u}{\partial x} = -xu^3 \Rightarrow x\frac{\partial u}{\partial x} = -x^2u^3 \qquad \text{...(1)}$$

Similarly $y\dfrac{\partial u}{\partial y} = -u^3y^2$ ...(2)

$$z\frac{\partial u}{\partial z} = -u^3z^2 \qquad \text{...(3)}$$

Adding (1), (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u^3(x^2 + y^2 + z^2) = -u^3 \cdot \frac{1}{u^2} = -u.$$

$\therefore \quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} + u = 0.$

This proves the first part.

Again, for the second part, we see that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(-u^3x)$$

$$= -\left\{u^3 \cdot 1 + x \cdot 3u^2\frac{\partial u}{\partial x}\right\} = -\left\{u^3 + 3xu^2(-u^3x)\right\}$$

$$= -u^3 + 3x^2u^5 \qquad \text{...(4)}$$

Similarly $\dfrac{\partial^2 u}{\partial y^2} = -u^3 + 3y^2u^5$ ...(5)

$$\frac{\partial^2 u}{\partial z^2} = -u^3 + 3z^2u^5 \qquad ...(6)$$

Adding (4), (5) and (6), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3u^3 + 3u^5(x^2 + y^2 + z^2)$$

$$= -3u^3 + 3u^5 \cdot \frac{1}{u^2} = -3u^3 + 3u^3 = 0.$$

Hence proved.

EXAMPLE 12. *If* $u = \log \sqrt{(x^2 + y^2 + z^2)}$, *then show that*

$$(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$

SOLUTION : Since $u = \log(x^2 + y^2 + z^2)^{1/2}$.

$$\therefore \qquad u = \tfrac{1}{2}\log(x^2 + y^2 + z^2).$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{2x}{x^2 + y^2 + z^2} = \frac{x}{x^2 + y^2 + z^2}. \qquad ...(1)$$

Hence $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{(x^2 + y^2 + z^2).1 - x.2x}{(x^2 + y^2 + z^2)^2} = \dfrac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}.$

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{z^2 + x^2 - y^2}{(x^2 + y^2 + z^2)^2}$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}.$$

Adding, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{(y^2 + z^2 - x^2) + (z^2 + x^2 - y^2) + (x^2 + y^2 - z^2)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}.$$

$$\therefore \quad (x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$

Note : Differentiating (1) partially w.r.t y.

$$\frac{\partial^2 u}{\partial x \partial y} = x \cdot \left\{-\frac{1}{(x^2 + y^2 + z^2)^2}\right\} \times 2y = \frac{-2xy}{(x^2 + y^2 + z^2)^2}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{-2xyz}{(x^2 + y^2 + z^2)^2}.$$

Therefore, in accordance with symmetry,

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y};\ \text{each} = \frac{-2xyz}{(x^2+y^2+z^2)^2}.$$

EXAMPLE 13. *If* $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *show that*

(i) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

(ii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

(B.U. 66H; R.U. 71H; P.U. 87; Kanpur 87)

SOLUTION : Given that $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}.$$

Adding, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{3}{x+y+z}.$$

Thus the first part is proved.

For the second part, we see that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= 3\left\{\frac{-1}{(x+y+z)^2}\right\} + 3\left\{\frac{-1}{(x+y+z)^2}\right\} + 3\left\{\frac{-1}{(x+y+z)^2}\right\}$$

$$= \frac{-9}{(x+y+z)^2}.$$

EXAMPLE 14. *If* $x^x y^y z^z = c$, *show that at* $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}. \qquad \textbf{(R.U. 68H)}$$

SOLUTION : Given that $x^x y^y z^z = c$.

Now taking logarithms of both sides,

$$x \log x + y \log y + z \log z = \log c$$

$$\therefore \quad z \log z = \log c - x \log x - y \log y \qquad \ldots(1)$$

Now differentiating (1) partially w.r.t. x, (treating y as const.)

$$\left(z \cdot \frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = -\left(x \cdot \frac{1}{x} + \log x\right)$$

$$\Rightarrow \quad (1 + \log z)\frac{\partial z}{\partial x} = -(1 + \log x)$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z}.$$

Similarly, from the symmetry,

$$\frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z}.$$

Now,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1 + \log y}{1 + \log z}\right)$$

$$= -(1 + \log y)\left\{-\frac{1}{(1 + \log z)^2}\right\} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{1 + \log y}{(1 + \log z)^2} \cdot \frac{1}{z} \cdot \left(-\frac{1 + \log x}{1 + \log z}\right)$$

putting the value of $\frac{\partial z}{\partial x}$

$$= -\frac{1}{z} \cdot \frac{(1 + \log x) + (1 + \log y)}{(1 + \log z)^3}.$$

Hence at the point $x = y = z$;

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x} \cdot \frac{(1+\log x)(1+\log x)}{(1+\log x)^3},$$

$$= -\frac{1}{x} \cdot \frac{1}{1+\log x} = -\frac{1}{x} \cdot \frac{1}{\log e + \log x}$$

$$= -\frac{1}{x} \cdot \frac{1}{\log ex} = -\frac{1}{x \log ex}$$

$$= -(x \log ex)^{-1}.$$

EXAMPLE 15. *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

(P.U. 49H, 53H, 57 H, 93H, 95H; B.U. 56H, 60 H, 66H; Bhag. 62H, 66H, 91H; M.U. 80H)

SOLUTION : Differentiating the given relation partially w.r.t. x (treating y and z constant)

$$\left\{\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2} \cdot \frac{\partial u}{\partial x}\right\} + \left\{-\frac{y^2}{(b^2+u)^2} \cdot \frac{\partial u}{\partial x}\right\} + \left\{-\frac{z^2}{(c^2+u)^2} \cdot \frac{\partial u}{\partial x}\right\} = 0.$$

$$\Rightarrow \quad \frac{2x}{a^2+u} - \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\} \frac{\partial u}{\partial x} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2x}{a^2+u} \Big/ \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\} \qquad \ldots(1)$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 = \frac{4x^2}{(a^2+u)^2} \Big/ \{ \quad " \quad \quad " \quad \quad "\}^2$$

Similarly $\left(\frac{\partial u}{\partial y}\right)^2 = \frac{4y^2}{(b^2+u)^2} \Big/ \{ \quad " \quad \quad " \quad \quad "\}^2$

$$\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4z^2}{(c^2+u)^2} \Big/ \{ \quad " \quad \quad " \quad \quad "\}^2$$

Adding, we get $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$

$$= 4 \Big/ \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\} \qquad \ldots(2)$$

Again, from (1), $x\frac{\partial u}{\partial x} = \frac{2x^2}{a^2+u} \Big/ \sum \frac{x^2}{(a^2+u)^2}$.

Similarly for $y\frac{\partial u}{\partial y}$ and $z\frac{\partial u}{\partial z}$.

Adding, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\sum \frac{x^2}{a^2+u} \Big/ \sum \frac{x^2}{(a^2+u)^2}$

$$= 2 \Big/ \sum \frac{x^2}{(a^2+u)^2} \qquad \ldots(3)$$

since from the given equation $\sum \frac{x^2}{a^2+u} = 1$.

Hence from (2) and (3),

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Example 7

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when

(i) $u = \tan^{-1}\frac{x}{y}$ (ii) $u = \log(x^2 + y^2)$

(iii) $u = \log(y\sin x + x\sin y)$ and verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(iv) If $u = x\cos y + y\cos x$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

2. If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$.

3. (i) If $x^3 + y^3 - x^3y^2z = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $x = y = 1$.

(P.U.57S; M. U. 91)

(ii) Find the value of $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$ when $r = ax^2 + 2hxy + by^2$. **(R.U. 85)**

4. If $u = \sin^{-1}\frac{x}{y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(R.U. 625)**

5. (i) If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

(ii) If $u = \sin\left(\sin\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(P.U. 64H)**

6. If $u = f(x^2 + y^2)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

7. If $u = x^2 + y^2 + z^2$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$.

8. If $u = \log(\tan x + \tan y + \tan z)$, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2. \qquad \textbf{(P.U. 46A; B.U. 75A)}$$

9. If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial y}{\partial z} = 0$.

10. If $u = x^2y + y^2z + z^2x$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$.

(B. U. 69A)

11. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$.

(Bhag. 71A; Mithila 79)

12. If $u = \frac{x^2 + y^2}{x + y}$, prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

13. If $u = a\sin\frac{x}{y} + b\cos\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(P.U. 41S, 46S)**

14. If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **(B.U. 63A)**

15. If $u = \log(x^2 + y^2)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. **(B.U. 57A)**

16. If $u = \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(B.U. 55S, 62A, 71A; P.U. 62A; R.U. 71A, 73A; Gorakhpur 87)

17. If $u = f(x + ay) + \phi(x - ay)$, prove that $\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$.

18. If $u = \tan(y + ax) + (y - ax)^{3/2}$, prove that $\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$.

19. If $u = e^x(x\cos x - y\sin y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

20. If $u = (x^2 + y^2)^{-1/2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^3$. **(P. U. 32A)**

21. If $u = 2(px + qy)^2 - (x^2 + y^2)$ and $p^2 + q^2 = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ **(M.U. 68A, 76H)**

22. If $u = \sqrt{x^2 + y^2 + z^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

(M.U. 72H; R.U. 67H)

23. If $u = \log(x^2 + y^2 + z^2)$, prove that $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$.

24. If $a^2x^2 + b^2y^2 - c^2z^2 = 0$, prove that $\frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2} \cdot \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}$.

25. If $u = \log r$ where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}.$$ **(Kanpur 84)**

26. If $u = \frac{x^2y^2}{x + y}$, show that $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}$.

27. If $u = \frac{\sin(ct - x)}{x}$, prove that $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x} \cdot \frac{\partial u}{\partial x}\right)$. **(P.U. 58A)**

28. If $u = e^{-mx}\sin(nt - mx)$, prove that $\frac{\partial u}{\partial t} = \frac{n}{2m^2} \cdot \frac{\partial^2 u}{\partial x^2}$. **(P.U. 58S)**

29. If $u = t^{-3/2} \cdot e^{-r^2/4kt}$, prove that $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial u}{\partial r}\right)$

(P.U. 57A; Kanpur 83)

30. If $\phi = (Ax^n + Bx^{-n})\cos n(y - \alpha)$, where A, B, n, α are constants, prove that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{x} \cdot \frac{\partial \phi}{\partial x} + \frac{1}{x^2} \cdot \frac{\partial^2 \phi}{\partial y^2} = 0.$$ **(P.U. 66A)**

31. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0$.

32. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

[**Hints :** Given $x = r\cos\theta$ and $y = r\sin\theta$ $\therefore$ $r^2 = x^2 + y^2$.

$\therefore$ Differentiating $2r\frac{\partial r}{\partial x} = 2x$ $\therefore$ $\frac{\partial r}{\partial x} = \frac{x}{r}$.

Again differentiating $\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$

$$= \frac{r - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - \frac{x^2}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}.$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3}$ etc.].

33. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$

except for $x = 0$, $y = 0$.

[Here $\tan\theta = \frac{y}{x}$ $\therefore$ $\theta = \tan^{-1}\frac{y}{x}$.]

34. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$; prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}.$$

35. If $z = xf(x + y) + yg(x + y)$, show that

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

36. Let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$$ **(B.U. 69H)**

[See Ex. 9 worked out]

37. If $u = f(r)$ where $x^2 = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}$$ **(R.U. 70H)**

ANSWERS

1. (i) $\frac{\partial u}{\partial x} = \frac{y}{x^2 + y^2}$, $\frac{\partial u}{\partial y} = \frac{-x}{x^2 + y^2}$ (ii) $\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$, $\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$

3. (i) $\frac{\partial z}{\partial x} = -3$, $\frac{\partial z}{\partial y} = -1$ (ii) $2(a + b)$.

3.3. Homogeneous Functions

Let $u = f(x, y)$ be any function of x and y. If the sum of the powers of x and y in each term of $f(x, y)$ be equal, then $f(x, y)$ is called a homogeneous function. In particular, if the sum of the powers of x and y in each term of $f(x, y)$ be 3, then $f(x, y)$ is called a homogeneous function of order 3; if the sum of the powers of x and y in each term of $f(x, y)$ be 4, then $f(x, y)$ is called a homogeneous function of order 4 and in general, if the sum of the powers of x and y in each term of $f(x, y)$ be n, then $f(x, y)$ is called a homogeneous function of order n. For example,

$x^3 + y^3 + 3x^2y$ is a homogeneous function of 3rd order;

$x^4 + 4x^2y^2 + y^4$ is a homogeneous function of 4th order, etc.

But $x^3 + y^3 + 3axy$ is not a homogeneous function, for the power of x in the first term x^3 is = 3, the power of y in the second term y^3 is = 3, but the sum of the powers of x and y in the third term $3axy$ is $=1 + 1 = 2$.

In general, if $f(x, y)$ is a hmogeneous function of nth order, then its form will be

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n \quad (1)$$

The sum of the powers of x and y in each term of (1) viz, the sum of the powers of x and y in the general term

$$a_rx^{n-r}y^r \text{ is } = (n - r) + r = n.$$

Therefore, $f(x, y)$ as given in (1) is a homogeneous function of nth order in x and y.

We can write the homogeneous function (1) in the following alternative way also.

$$f(x, y) = x^n\left[a_0 + \frac{a_1x^{n-1}y}{x^n} + \frac{a_2x^{n-2}y^2}{x^n} + \ldots + \frac{a_ny^n}{x^n}\right]$$

$$= x^n\left[a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \ldots + a_n\frac{y^n}{x^n}\right]$$

$$= x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right]$$

$$= x^n\phi\left(\frac{y}{x}\right) \text{ where } \phi \text{ is a function of } \frac{y}{x}.$$

Hence if $f(x, y)$ be a homogeneous function of nth order, then we can write $f(x, y)$ as

$$f(x, y) = x^n\phi\left(\frac{y}{x}\right).$$

Note : The power of x in the first factor x^n denotes the fact as to which order $f(x, y)$ is a homogeneous function. Thus if $f(x, y)$ is a homogeneous function of 2nd order, then

$$f(x,y) = x^2\phi\left(\frac{y}{x}\right).$$

If $f(x,y)$ is a homogeneous function of 3rd order, then

$$f(x, y) = x^3 \phi\left(\frac{y}{x}\right) \text{ and so on.}$$

Second criterion : Let us suppose that

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n.$$

Putting tx for x and ty for y, we have

$$f(tx, ty) = t^n[a_0x^n + a_1x^{n-1}y + ... + a_ny^n]$$

$$= t^nf(x, y).$$

Hence if $f(x, y)$ is a homogeneous function of nth order, then

$$f(tx, ty) = t^nf(x, y),$$

Similarly if $f(x, y, z)$ is a homogeneous function of nth order, then

$$f(tx, ty, tz) = t^nf(x, y, z) \text{ etc.}$$

3.4. Euler's Theorem on Homogeneous function of two variables:

Statement : *If $u = f(x, y)$ be a homogeneous function in x and y of degree n, then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof : Let $u = f(x, y)$

$$= A_1x^{\alpha_1}y^{\beta_1} + A_2x^{\alpha_2}y^{\beta_2} + A_3x^{\alpha_3}y^{\beta_3} + ... + A_nx^{\alpha_n}y^{\beta_n} \quad ...(1)$$

where $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + \beta_3 = ... = \alpha_n + \beta_n = n$.

Now differentiating (1) partially w.r.t. x, we have

$$\frac{\partial u}{\partial x} = A_1\alpha_1x^{\alpha_1-1}y^{\beta_1} + A_2\alpha_2x^{\alpha_2-1}y^{\beta_2}$$

$$+ A_3\alpha_3x^{\alpha_3-1}y^{\beta_3} + ... + A_n\alpha_nx^{\alpha_n-1}y^{\beta_n}$$

$$\therefore x\frac{\partial u}{\partial x} = A_1\alpha_1x^{\alpha_1}y^{\beta_1} + A_2\alpha_2x^{\alpha_2}y^{\beta_2} + ... + A_n\alpha_nx^{\alpha_n}y^{\beta_n}. \quad ...(2)$$

Similarly differentiating (1) partially w.r.t. y and then multiplying by y, we have

$$y\frac{\partial u}{\partial y} = A_1\beta_1x^{\alpha_1}y^{\beta_1} + A_2\beta_2x^{\alpha_2}y^{\beta_2} + ... + A_n\beta_nx^{\alpha_n}y^{\beta_n} \quad ...(3)$$

Now adding (2) and (3),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (\alpha_1 + \beta_1)A_1x^{\alpha_1}y^{\beta_1} + (\alpha_2 + \beta_2)A_2x^{\alpha_2}y^{\beta_2}$$

$$+ (\alpha_3 + \beta_3)A_3x^{\alpha_3}y^{\beta_3} + ... + (\alpha_n + \beta_n)A_nx^{\alpha_n}y^{\beta_n}$$

$$= n[A_1x^{\alpha_1}y^{\beta_1} + A_2x^{\alpha_2}y^{\beta_2} + ... + A_nx^{\alpha_n}y^{\beta_n}]$$

$$\because \quad \alpha_1 + \beta_1 = \alpha_2 + \beta_2 = ... = \alpha_n + \beta_n; \text{ each } = n$$

$$= nu.$$

Alternative proof :

Since u is a homogeneous function of the nth degree in x and y, therefore we can write u as

$$u = x^n f\left(\frac{y}{x}\right) \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -x^{n-1}y f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right). \qquad ...(2)$$

Again differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\therefore \quad y\frac{\partial u}{\partial y} = x^{n-1}y f'\left(\frac{y}{x}\right) \qquad ...(3)$$

Now, adding (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

Note : For another proof of Euler's Theorem, see page 146.

Generalisation :

If $u = f(x, y, z)$ be a homogeneous function of order n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

As before, let

$$u = f(x, y, z)$$

$$= A_1x^{\alpha_1}y^{\beta_1}z^{\gamma_1} + A_2x^{\alpha_2}y^{\beta_2}z^{\gamma_2} + ... + A_nx^{\alpha_n}y^{\beta_n}z^{\gamma_n}$$

where $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = ...= n.$

Now, differentiating partially w.r.t. x, y, z successively, it can be shown as before that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial z} + z\frac{\partial u}{\partial z} = nu.$$

Obs. For another method, which may be useful in the case of any number of variables, see the section on total differential (page 144).

Worked Out Examples

EXAMPLE 1. *Verify Euler's Theorem when* $u = x^3\log\frac{y}{x}$. **(P.U. 28A; M.U. 66A)**

SOLUTION : Here putting tx for x and ty for y, we get

$$u(tx, ty) = t^3x^3\log\frac{ty}{tx} = t^3x^3\log\frac{y}{x} = t^3u(x, y).$$

$\therefore$ u is a homogeneous function of order 3.

From the given relation,

$$u = x^3(\log y - \log x).$$

$$\therefore \quad \frac{\partial u}{\partial x} = x^3\left(0 - \frac{1}{x}\right) + (\log y - \log x)3x^2, \text{ (treating } y \text{ as const.)}$$

$$= -x^2 + 3x^2(\log y - \log x)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -x^3 + 3x^3(\log y - \log x) \qquad ...(1)$$

$$\text{Again } \frac{\partial u}{\partial y} = x^3\left(\frac{1}{y} - 0\right), \text{ (treating } x \text{ as const.)}$$

$$= x^3/y$$

$$\therefore \quad y\frac{\partial u}{\partial y} = x^3 \qquad ...(2)$$

Now, adding (1) and (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3(\log y - \log x) = 3u.$$

But here the given function $u = x^3\log\frac{y}{x}$ is a homogeneous function of 3rd order.

Thus the given function satisfies Euler's Theorem.

EXAMPLE 2. *Verify Euler's Theorem for the expression* $x^n\sin\frac{y}{x}$.

(B.U. 58S, 64S; Bhag.U. 62S; P.U. 50A, 62A; R.U. 68A, 72S; M.U. 76)

SOLUTION : Let $u = x^n \sin\frac{y}{x}$.

$$\therefore \quad \frac{\partial u}{\partial x} = x^n \times \cos\frac{y}{x} \cdot y\left(-\frac{1}{x^2}\right) + \sin\frac{y}{x} \times nx^{n-1}; \text{ (treating } y \text{ as const.)}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -x^{n-1}y\cos\frac{y}{x} + nx^n\sin\frac{y}{x} \qquad ...(1)$$

Again, $\frac{\partial u}{\partial y} = x^n\cos\frac{y}{x} \cdot \frac{1}{x}$; (treating x as const.)

$$\therefore \quad y\frac{\partial u}{\partial y} = x^{n-1}y\cos\frac{y}{x} \qquad ...(2)$$

Now, adding (1) and (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\sin\frac{y}{x} = nu.$$

But here the given function $x^n\sin\frac{y}{x}$ is a homogeneous function of nth order. Thus the given function satisfies Euler's Theorem.

EXAMPLE 3. *Verify Euler's Theorem when* $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$.

(B.U. 54S; P.U. 37A, 64S)

SOLUTION : Here the given function

$$u = \frac{x.x^3(1 - y^3/x^3)}{x^3(1 + y^3/x^3)} = x.\frac{1 - y^3/x^3}{1 + y^3/x^3} = x\,\phi\left(\frac{y}{x}\right)$$

i.e. it is a homogeneous function of 1st order.

Taking logarithm of both sides of the given function, we get

$$\log u = \log x + \log(x^3 - y^3) - \log(x^3 + y^3) \qquad ...(1)$$

Now differentiating both sides of (1) partially w.r.t. x (treating y as const.)

$$\frac{1}{u}\cdot\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3 - y^3} - \frac{3x^2}{x^3 + y^3}$$

$$\therefore \quad \frac{1}{u}\cdot x\frac{\partial u}{\partial x} = 1 + \frac{3x^3}{x^3 - y^3} - \frac{3x^3}{x^3 + y^3} \qquad ...(2)$$

Similarly differentiating both sides of (1) partially w. r. to y (treating x as constant.)

$$\frac{1}{u}\cdot y\frac{\partial u}{\partial y} = 0 + \frac{-3y^2}{x^3 - y^3} - \frac{3y^2}{x^3 + y^3}$$

$$\therefore \quad \frac{1}{u}\cdot y\frac{\partial u}{\partial y} = \frac{-3y^3}{x^3 - y^3} - \frac{3y^3}{x^3 + y^3} \qquad ...(3)$$

Now, adding (2) and (3),

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$$

$$=1+3-3=1.$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u.$$

Thus Euler's Theorem is satisfied.

EXAMPLE 4. *If* $u=\log\dfrac{x^2+y^2}{x+y}$, *prove that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1$.

(B.U. 62S; P.U. 56S, 70A; M.U. 77, 92)

SOLUTION : From the given equation,

$$e^u=\frac{x^2+y^2}{x+y}=\frac{x^2\left\{1+\left(\frac{y}{x}\right)^2\right\}}{x\left(1+\frac{y}{x}\right)}=x\phi\left(\frac{y}{x}\right).$$

Therefore e^u is a homogeneous function of first (1) order in x and y.

Let $e^u=v$.

$\therefore$ According to Euler's Theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1.v. \qquad \ldots(1)$$

But from $v=e^u$,

$$\frac{\partial v}{\partial x}=e^u.\frac{\partial u}{\partial x};\quad \frac{\partial v}{\partial y}=e^u\cdot\frac{\partial u}{\partial y}.$$

$\therefore$ from (1),

$$x\cdot e^u\frac{\partial u}{\partial x}+y\cdot e^u\cdot\frac{\partial u}{\partial y}=e^u.$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.$$

EXAMPLE 5. *If* $u=\cos^{-1}\dfrac{x-y}{x+y}$, *prove that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

(R.U. 66A, 80; B.U. 55A; M.U. 65A; Bhag. 68A)

SOLUTION : From the given equation,

$$\cos u=\frac{x-y}{x+y}=\frac{x\left(1-\frac{y}{x}\right)}{x\left(1+\frac{y}{x}\right)}=\phi\left(\frac{y}{x}\right).$$

$\therefore$ $\cos u$ is a homogeneous function of zero (0) order.

Let $\cos u = v$.

$\therefore$ According to Euler's Theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0. \quad v = 0. \qquad \ldots(1)$$

But from $v = \cos u$, $\dfrac{\partial v}{\partial x} = -\sin u \dfrac{\partial u}{\partial x}$,

$$\frac{\partial v}{\partial y} = -\sin u \frac{\partial u}{\partial y}.$$

$\therefore$ from (1),

$$x(-\sin u)\frac{\partial u}{\partial x} + y\left(-\sin u \frac{\partial u}{\partial y}\right) = 0$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

EXAMPLE 6. *If* $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

(R.U. 66A, 68S, 71A, 81; B.U. 56S, 67A; P.U. 56A, 62S, 68A, 76A; Bhag. 69A, 86, 89H; M.U. 72S, 74A, 78H, 82H, 87, 91H, 93H; Kanpur 84)

SOLUTION : From the given equation,

$$\sin u = \frac{x^2 + y^2}{x + y} = \frac{x^2\left(1 + \dfrac{y^2}{x^2}\right)}{x\left(1 + \dfrac{y}{x}\right)}.$$

$\therefore$ $\sin u$ is a homogeneous function of first order (order 1) in x and y.

Let $v = \sin u$.

Therefore v is a homogeneous function of order 1 in x and y.

Hence according to Euler's Theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1.v. \qquad \ldots(1)$$

But since $v = \sin u$

$$\therefore \quad \frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}; \quad \frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}.$$

Therefore from (1),

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u; \text{ on dividing by } \cos u.$$

Hence the result.

EXAMPLE 7. *If* $u = \sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, *show that* $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\cdot\dfrac{\partial u}{\partial y}$.

(Bhag. 63A; P.U. 69S, 72A)

SOLUTION : From the given equation,

$$v = \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}.$$

Putting tx for t and ty for y, we have

$$v(tx, ty) = \frac{\sqrt{tx}-\sqrt{ty}}{\sqrt{tx}+\sqrt{ty}} = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = t^0\, v(x, y).$$

$\therefore$ $\sin u = v$ is a homogeneous function of zero order.

Therefore according to Euler's Theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0. \quad v = 0. \qquad \ldots(1)$$

But $\because v = \sin u \quad \therefore \dfrac{\partial v}{\partial x} = \cos u\dfrac{\partial u}{\partial x}$

$$\text{and } \frac{\partial v}{\partial y} = \cos u\frac{\partial u}{\partial y}.$$

Hence from (1),

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0.$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad x\frac{\partial u}{\partial x} = -y\frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{y}{x}\cdot\frac{\partial u}{\partial y}.$$

EXAMPLE 8. *If* $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

(Bhag.U. 66A, 94; P.U. 55A, 64A, 66H, 73A; B.U. 65H, 70A, 74A; R.U. 84; Kanpur 88)

SOLUTION : From the given equation,

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\dfrac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$= x^{\frac{1}{2}}\,\phi\left(\frac{y}{x}\right).$$

Hence $\sin u$ is a homogeneous function of order $\frac{1}{2}$ in x and y.

Let $v = \sin u$.

$\therefore$ According to Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v \qquad ...(1)$$

But from $v = \sin u$, we have

$$\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}; \quad \frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}.$$

$\because$ from (1),

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u.$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

EXAMPLE 9. *If* $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

(Bhag. 65A, 65H, 69S, 85, 91; B.U. 59A, 65A, 86; P.U. 56H, 61A, 68S, 69A; R.U. 67S, 76A; M.U. 90; Gorakhpur 82; Kanpur 85)

SOLUTION : From the given equation,

$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3\left(1 + \dfrac{y^3}{x^3}\right)}{x\left(1 - \dfrac{y}{x}\right)} = x^2\phi\left(\frac{y}{x}\right).$$

$\therefore$ $\tan u$ is a homogeneous function of order 2 in x and y.

Let $v = \tan u$.

Therefore according to Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v. \qquad ...(1)$$

But from $v = \tan u$, we have

$$\frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}.$$

$\therefore$ from (1),

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u.$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= \frac{2\sin u \times \cos^2 u}{\cos u} = 2\sin u\cos u = \sin 2u.$$

Hence the result.

EXAMPLE 10. *If* $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

(R.U. 67A, 69A, 72A; B.U. 63A, 69S; Bhag. 65S, 66S; M.U. 67A, 71A, 79, 89H)

SOLUTION : First we transform $\sin^{-1}\frac{x}{y}$ in the form of $\tan^{-1}\frac{x}{y}$.

Let $\theta = \sin^{-1}\frac{x}{y}$, then $\sin\theta = \frac{x}{y}$.

$\therefore \quad \tan\theta = \frac{x}{\sqrt{y^2 - x^2}}$ i.e. $\theta = \tan^{-1}\frac{x}{\sqrt{y^2 - x^2}}$.

Thus $u = \tan^{-1}\frac{x}{\sqrt{y^2 - x^2}} + \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\frac{x}{\sqrt{y^2 - x^2}} + \frac{y}{x}}{1 - \frac{x}{\sqrt{y^2 - x^2}} \cdot \frac{y}{x}}$

$$= \tan^{-1}\frac{x^2 + y\sqrt{y^2 - x^2}}{x\sqrt{y^2 - x^2} - xy}.$$

$$\Rightarrow \tan u = \frac{x^2 + y\sqrt{y^2 - x^2}}{x\sqrt{y^2 - x^2} - xy}$$

$$= \frac{x^2\left\{1 + \frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}\right\}}{x^2\left\{\sqrt{\left(\frac{y}{x}\right)^2 - 1} - \frac{y}{x}\right\}} = \phi\left(\frac{y}{x}\right).$$

$\therefore$ $\tan u$ is a homogeneous function of zero (0) order.

This can also be verified by writing tx for x and ty for y in the R.H.S.

Let $v = \tan u$.

$\therefore$ According to Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.v = 0 \qquad \ldots(1)$$

But since $v = \tan u$, therefore

$$\frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}.$$

$\therefore$ from (1),

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} + 0$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Note : This problem has also been worked out before by direct clacula-tion.

Example 8

1. Verify Euler's Theorem for the functions :

(i) $u = ax^2 + 2hxy + by^2$ (ii) $u = \dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$

2. If $\sin u = \dfrac{x^2 - y^2}{x^2 + y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

3. If $u = \tan^{-1}\dfrac{x^2 + y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u.$ **(P.U. 59A; Bhag. 88)**

4. If $u = \sec^{-1}\dfrac{x^2 + y^2}{x - y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \cot u.$ **(P.U. 65A)**

5. If $u = \sec^{-1}\dfrac{x^3 + y^3}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u.$

6. If $\sin u = \dfrac{x^3 + y^3}{x - y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u.$

(P.U. 60S, 63S, 70S; M.U. 85)

7. If $u = \cos^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0.$

(Bhag. 63S, 66A, 92A; B.U. 72A; M.U. 82A)

8. If $u = xy f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u.$

9. If $u = \sin(\sqrt{x} + \sqrt{y})$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}(\sqrt{x} + \sqrt{y})\cos(\sqrt{x} + \sqrt{y}).$

(B.U. 64A)

10. If $u = \sin\sqrt{\dfrac{x - y}{x + y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

(Euler's formula holds good) **(P.U. 38)**

11. Given that $F(u) = V(x, y, z)$ where V is a homogeneous function in x, y, z of degree n, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)}.$$

12. If H be a homogeneous function of degree n in x and y and if $u = (x^2 + y^2)^{-n/2}$, show that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) = 0. \qquad \textbf{(P.U. 59A; R.U. 70S)}$$

[**Hint :** $\dfrac{\partial}{\partial x}\left(H\dfrac{\partial u}{\partial x}\right) = H\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}\dfrac{\partial H}{\partial x}$

$$\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) = H\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot\frac{\partial H}{\partial y}.$$

Since H is a homogeneous function of degree n in x and y, therefore

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} = nH.]$$

3.5. Theorem : *If $u = f(x, y)$ is a homogeneous function of degree n, then*

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u.$ **(M.U. 75H; P.U. 88)**

Proof. Since $u = f(x, y)$ is a homogeneous function of nth order, therefore from Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \qquad \ldots(1)$$

Differentiating (1) partially w.r.t x (treating y as const.)

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot 1 + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = (n-1)\frac{\partial u}{\partial x} \qquad \ldots(2)$$

This proves (i).

Again, differentiating (1) partially w.r.t. y (treating x as const.)

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial^2 y} = (n-1)\frac{\partial u}{\partial y} \qquad \ldots(3)$$

This proves (ii).

Now multiplying (2) by x and (3) by y and then adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= (n-1)nu = n(n-1)u.$$

Thus (iii) is proved. This result should be remembered as a formula.

Second Method :

Treating $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ as an operator, equivalent to D, we shall have

$$D(u) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu;$$

from Euler's Theorem

$$D^2u = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(nu)$$

$$= n\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = n.nu = n^2u \qquad \ldots(1)$$

But $D^2u = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

$$= x\frac{\partial}{\partial x}\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} + y\frac{\partial}{\partial y}\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\}$$

$$= x\left[\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + \left(y\frac{\partial^2 u}{\partial x \partial y}\right)\right] + y\left[\left(x\frac{\partial^2 u}{\partial y \partial x}\right) + \left(y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\right)\right]$$

$$= \left[x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y}\right] + \left[xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}\right]$$

$$= \left\{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right\} + \left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\}$$

$$= \left\{ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right\} + nu \qquad \ldots(2)$$

From (1) and (2),

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n^2 u$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n^2 - n)u = n(n-1)\, u.$$

Cor. It follows that if u be a homogeneous function of order 1, (or zero) then,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

EXAMPLE 1. *If* $u = \dfrac{(x^2 + y^2)^m}{2m(2m-1)} + x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, *prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)^m.$$

(B.U. 63H, 67H, 71H; M.U. 64H)

SOLUTION : Let $X = \dfrac{(x^2 + y^2)^m}{2m(2m-1)}$, $Y = x\phi\left(\dfrac{y}{x}\right)$ and $Z = \psi\left(\dfrac{y}{x}\right)$.

Then $u = X + Y + Z$.

We can write $(x^2 + y^2)^m$ as $x^{2m}\left(1 + \dfrac{y^2}{x^2}\right)^m$ and hence X is a homogeneous function of $(2m)$th degree. Therefore by Euler's Theorem

$$x^2 \frac{\partial^2 X}{\partial x^2} + 2xy \frac{\partial^2 X}{\partial x \partial y} + y^2 \frac{\partial^2 X}{\partial y^2} = 2m(2m-1)X \qquad \ldots(1)$$

Also, obviously Y and Z are homogeneous functions of 1st degree and 0 (zero) degree respectively.

Hence by Euler's Theorem, again

$$x^2 \frac{\partial^2 Y}{\partial x^2} + 2xy \frac{\partial^2 Y}{\partial x \partial y} + y^2 \frac{\partial^2 Y}{\partial y^2} = 1(1-1)Y$$

$$= 0 \qquad \ldots(2)$$

and

$$x^2 \frac{\partial^2 Z}{\partial x^2} + 2xy \frac{\partial^2 Z}{\partial x \partial y} + y^2 \frac{\partial^2 Z}{\partial y^2} = 0(0-1)Z$$

$$= 0 \qquad \ldots(3)$$

Since $u = X + Y + Z$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Z}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 X}{\partial x \partial y} + \frac{\partial^2 Y}{\partial x \partial y} + \frac{\partial^2 Z}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 Y}{\partial y^2} + \frac{\partial^2 Z}{\partial y^2}.$$

Now adding (1), (2) and (3) and using the above equations, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2m(2m-1)X$$

$$= 2m(2m-1)\frac{(x^2+y^2)^m}{2m(2m-1)} = (x^2+y^2)^m$$

EXAMPLE 2. *If* $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, *find the value of*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

SOLUTION : From the given relation,

$$\tan u = \frac{x^3+y^3}{x-y} = v \text{ (say)}.$$

This is obvious that v is a homogeneous function of 2nd order.

Therefore from the formula,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 2(2-1)v = 2v. \qquad \ldots(1)$$

But $\because v = \tan u$, therefore

$$\frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \sec^2 u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot 2\sec^2 u\tan u\cdot\frac{\partial u}{\partial x}$$

$$= \sec^2 u\frac{\partial^2 u}{\partial x^2} + 2\sec^2 u\tan u\left(\frac{\partial u}{\partial x}\right)^2.$$

Similarly $\quad \dfrac{\partial^2 v}{\partial y^2} = \sec^2 u\dfrac{\partial^2 u}{\partial y^2} + 2\sec^2 u\tan u\left(\dfrac{\partial u}{\partial y}\right)^2.$

Again, $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(\sec^2 u \dfrac{\partial u}{\partial x}\right)$

$$= \sec^2 u \cdot \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} 2\sec^2 u \tan u \frac{\partial u}{\partial y}$$

$$= \sec^2 u \cdot \frac{\partial^2 u}{\partial x \partial y} + 2 \sec^2 u \tan u \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right).$$

Hence from (1),

$$x^2\left\{\sec^2 u \frac{\partial^2 u}{\partial x^2} + 2\sec^2 u \tan u \left(\frac{\partial u}{\partial x}\right)^2\right\} + 2xy\left\{\sec^2 u \frac{\partial^2 u}{\partial x \partial y} + 2\sec^2 u \tan u \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right\}$$

$$+ y^2\left\{\sec^2 u \frac{\partial^2 u}{\partial y^2} + 2\sec^2 u \tan u \left(\frac{\partial u}{\partial y}\right)^2\right\} = 2\tan u.$$

$$\Rightarrow \sec^2 u\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right)$$

$$+ 2\sec^2 u \tan u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + y^2\left(\frac{\partial u}{\partial y}\right)^2\right] = 2\tan u. \quad \ldots(2)$$

Now $x^2\left(\dfrac{\partial u}{\partial x}\right)^2 + 2xy\left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right) + y^2\left(\dfrac{\partial u}{\partial y}\right)^2 = \left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right)^2$

$= (\sin 2u)^2$; Ex. 9. page 130

Hence from (2),

$$\sec^2 u\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right) + 2\sec^2 u.\tan u.\sin^2 2u = 2\tan u$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2\tan u(1 - 2\sec^2 u \sin^2 2u)}{\sec^2 u}$$

$$= \sin 2u(1 - 4\sin^2 u).$$

EXAMPLE 3. *If* $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$, *show that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

(B.U. 57H, 59H; Bhag. 63H, 67H, 93H; M.U. 70H, 92H; P.U. 86; Gorakhpur 83)

SOLUTION : From the given relation, we have

$$\operatorname{cosec} u = \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$$

$$\Rightarrow \quad \operatorname{cosec}^2 u = \frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}$$

$$\Rightarrow \quad \sin^2 u = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}.$$

Let $\sin^2 u = v$. Then

$$v = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}.$$

From above, it follows that v is a homogeneous function of x and y and is of degree $\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$.

Hence $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -\frac{1}{6}v = -\frac{1}{6}\sin^2 u.$ ...(1)

Also, $x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$

$$= -\frac{1}{6}\left(-\frac{1}{6} - 1\right)\sin^2 u = \frac{7}{36}\sin^2 u. \qquad \ldots(2)$$

Now, $v = \sin^2 u$. Therefore

$$\frac{\partial v}{\partial x} = 2\sin u \cos u\frac{\partial u}{\partial x} = \sin 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \sin 2u\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 v}{\partial x^2} = 2\cos 2u\left(\frac{\partial u}{\partial x}\right)^2 + \sin 2u\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial y^2} = 2\cos 2u\left(\frac{\partial u}{\partial y}\right)^2 + \sin 2u\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\sin 2u\frac{\partial u}{\partial x}\right)$$

$$= \sin 2u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} \cdot 2\cos 2u\frac{\partial u}{\partial y}$$

$$= \sin 2u\frac{\partial^2 u}{\partial x \partial y} + 2\cos 2u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)$$

Hence (1) becomes

$$x.\sin 2u\frac{\partial u}{\partial x} + y\sin 2u\frac{\partial u}{\partial y} = -\frac{1}{6}\sin^2 u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}\frac{\sin^2 u}{\sin 2u}$$

$$= -\frac{1}{6}\cdot\frac{\sin^2 u}{2\sin u\cos u} = -\frac{1}{12}\tan u \qquad \ldots(3)$$

Also, from (2), we get

$$x^2\left\{2\cos 2u\left(\frac{\partial u}{\partial x}\right)^2 + \sin 2u\frac{\partial^2 u}{\partial x^2}\right\} + 2xy\left\{\sin 2u\frac{\partial^2 u}{\partial x\partial y} + 2\cos 2u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right\}$$

$$+ y^2\left\{2\cos 2u\left(\frac{\partial u}{\partial y}\right)^2 + \sin 2u\frac{\partial^2 u}{\partial y^2}\right\} = \frac{7}{36}\sin^2 u$$

$$\Rightarrow \quad \left\{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right\}\sin 2u$$

$$+ 2\cos 2u\left\{x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + y^2\left(\frac{\partial u}{\partial y}\right)^2\right\}$$

$$= \frac{7}{36}\sin^2 u$$

$$\Rightarrow \quad \left\{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right\}\sin 2u$$

$$= \frac{7}{36}\sin^2 u - 2\cos 2u\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\}^2$$

$$= \frac{7}{36}\sin^2 u - 2\cos 2u\left\{-\frac{1}{12}\tan u\right\}^2; \quad \text{from (3)}$$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{7}{36}\frac{\sin^2 u}{\sin 2u} - \frac{2\cos 2u}{\sin 2u}\cdot\frac{1}{144}\tan^2 u$$

$$= \frac{7}{36}\cdot\frac{\sin^2 u}{2\sin u\cos u} - \frac{2}{\tan 2u}\cdot\frac{1}{144}\tan^2 u$$

$$= \frac{7}{72}\tan u - \frac{1}{72}\cdot\frac{1-\tan^2 u}{2\tan u}\cdot\tan^2 u$$

$$= \frac{1}{72}\tan u\left\{7 - \frac{1-\tan^2 u}{2}\right\}$$

$$= \frac{1}{72}\tan u\left\{\frac{13+\tan^2 u}{2}\right\}$$

Hence the result.

EXAMPLE 4. *If w denotes the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \ldots\ldots$ and if u be a homogeneous function of n dimensions in the variables $x, y, z, \ldots\ldots$, show that*

$$w(w-1)(w-2)\ldots\ldots(w-r)\log u = (-1)^r n.r! \quad \textbf{(M. U. 63H; Bhag. 70H)}$$

SOLUTION : We first calculate $(w-r)\log u$.

For this, $(w-r)\log u = w\log u - r\log u$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \ldots.\right)\log u - r\log u$$

$$= \left\{x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) + \ldots.\right\} - r\log u$$

$$= \left\{x\cdot\frac{1}{u}\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\frac{\partial u}{\partial y} + \ldots\right\} - r\log u$$

$$= \frac{1}{u}\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots\right\} - r\log u$$

$$= \frac{1}{u}\cdot nu - r\log u;$$

since u is a homogeneous function of order n.

$$= n - r\log u.$$

$\therefore\ w(w-1)(w-2)\ldots(w-r)\log u$

$$= w(w-1)(w-2)\ldots.(w-r+1)(n-r\log u)$$

$$= w(w-1)(w-2)\ldots.(w-r+1)n + (-r)\,w(w-1)(w-2)\ldots.(w-r+1)r\log u$$

$$= E_1 + E_2 \text{ (say)} \qquad \ldots(1)$$

But $E_1 = w(w-1)(w-2)\ldots.(w-r+1)n$

$= w[(w-1)(w-2)\ldots.\{-(r-1)n\}]$

$= w[(-r+1)(-r+2)\ldots.(-1)] = w(\text{const}) = 0.$

$E_2 = (-r)w(w-1)(w-2)\ldots.(w-r+2)(-r+1)\log u$

$= (-r)(-r+1)(-r+2) + \ldots.(-1)\,w\log u$

$= (-1)^r r!\,\frac{1}{u}nu = (-1)^r.n.r!$

Hence the result.

Example 9

1. If $u = \dfrac{x^2y^2}{x^2+y^2}$, show that

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial u}{\partial x}$ (ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial y}$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2u.$

2. If $u = \dfrac{x^3+y^3}{x^2+y^2}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$ **(Bhag. 92H)**

3. If $u = (x^2+y^2)^{2/3}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{4}{9}u.$

4. If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \tan^3 u.$

(B. U. 92H)

5. If $u = x^{n-1}yf\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u.$

6. If $u = x\phi\left(\dfrac{x}{y}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

(P.U. 47H, 60H; M.U. 69H; B.U. 90H)

3.6. Total Differential

Let $u = f(x, y)$ be a function of two variables x and y.

Let us suppose that u takes up the value $u + \delta u$ when x takes up the value $x + \delta x$ and y, $y + \delta y$ so that when $\delta x, \delta y \to 0$, $\delta u \to 0$.

$\therefore \quad u + \delta u = f(x + \delta x, y + \delta y).$

On subtraction, $\delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y);$

by adding and subtracting $f(x, y + \delta y)$ at the same time.

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \delta x + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \delta y \quad ...(1)$$

By Mean-value Theorem, we have

$$\frac{f(x + \delta x, y + \delta y) - f(x y + \delta y)}{\delta x} = \frac{\partial}{\partial x} f(x + \theta_1 \delta x, y + \delta y)$$

where $0 < \theta_1 < 1$.

and $$\frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial}{\partial y} f(x, y + \theta_2 \delta y)$$ where $0 < \theta_2 < 1$.

Substituting these values in (1), we get

$$\delta u = \frac{\partial}{\partial x} f(x + \theta_1 \delta x, y + \delta y)\delta x + \frac{\partial}{\partial y}(f, y + \theta_2 \delta y)\delta y. \quad ...(2)$$

We suppose that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in the neighbourhood of the point (x, y). Therefore, we may write

$$\frac{\partial}{\partial x} f(x + \theta_1 \delta x, y + \delta y) = \frac{\partial}{\partial x} f(x, y) + \varepsilon_1$$

and

$$\frac{\partial}{\partial y} f(x, y + \theta_2 \delta y) = \frac{\partial}{\partial y} f(x, y) + \varepsilon_2$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\delta x, \delta y \to 0$.

Hence from (2), we get

$$\delta u = \frac{\partial}{\partial x} f(x, y)\delta x + \frac{\partial}{\partial y} f(x, y)\delta y + \varepsilon_1 \delta x + \varepsilon_2 \delta y. \quad ...(3)$$

Therefore from (3), in the limiting stage when $\delta x \to 0$, $\delta y \to 0$, $\delta u \to 0$, we find that

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

i.e.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Note : If $u = f(x, y, z)$ is a function of three variables, then it can be proved similarly by writing

$$\begin{aligned}\delta u &= f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z) \\ &= \{f(x + \delta x, y + \delta y, z + \delta z) - f(x, y + \delta y, z + \delta z)\} \\ &\quad + \{f(x, y + \delta y, z + \delta z) - f(x, y, z + \delta z)\} \\ &\quad + \{f(x, y, z + \delta z) - f(x, y, z)\}\end{aligned}$$

or it can be proved by the use of Taylor's Theorem (about which we shall study in higher classes) that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

3.7. Exact Differential

Definition : *If an expression of the type $Mdx + Ndy$ (M and N are functions of x and y or constant) may be reduced to du, where u is a function of x and y, then $Mdx + Ndy$ is said to be an exact differential.*

Theorem : *The necessary and sufficient condition that the expression* $Mdx + Ndy$ *be an exact differential is* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

***Proof* : (i) The condition is necessary.**

Let the given expression $Mdx + Ndy$ be an exact differential and let it be equal to du.

Now $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

i.e. $$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

On comparison, we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Since $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(ii) The condition is sufficient.

In this case, we are given that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; and we are to prove that $Mdx + Ndy = du$.

Let $\int Mdx = P \quad \therefore \quad M = \frac{\partial P}{\partial x}$.

Now, $$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$$

$\Rightarrow$ $$N = \frac{\partial P}{\partial y} + \phi(y) \text{ where } \phi(y) \text{ is a function of } y$$

Therefore we have

$$Mdx + Ndy = \frac{\partial P}{\partial x} \cdot dx + \frac{\partial P}{\partial y}dy + \phi(y)dy$$

$$= d[P + \psi(y)] \; ; \quad \text{where } d[\psi(y)] = \phi(y)dy$$

Now, let $P + \psi(y) = u$ where u is a function of x and y

$\therefore \quad Mdx + Ndy = du.$

Hence the result.

EXAMPLE *Prove that the expression* $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy$ *is an exact differential.*

SOLUTION : Here $M = x^3 + 3xy^2$ and $N = y^3 + 3x^2y$.

$$\therefore \quad \frac{\partial M}{\partial y} = 3x.2y = 6xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 3.2x.y = 6xy$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which proves the result.

Example 10

1. Prove that the following expressions are exact differential :

(i) $(x^2 + y)dx + (x + e^y)dy$

(ii) $\sin x \sin y dx - \cos x \cos y dy$

(iii) $ye^x dx + e^x dy$

(iv) $(y - 2x - 1)dx - (2y - x - 1)dy$

(v) $(2x + y - 3)dx + (x - 4y + 1)dy$ **(Bhag. 90)**

(vi) $(4x + 3y - 4)dx + (3x - 7y - 3)dy$

2. Prove that $(x^2 + y^2)dx - 2xydy$ is not an exact differential.

3. Show that the expression $ydx - xdy$ is not an exact differential but $\frac{1}{y^2}(ydx - xdy)$ is an exact differential.

4. If $Pdx + Qdy + Rdz$ can be made a perfect differential of some functions of x, y, z by multiplying each term by a common factor, show that,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad \textbf{(P.U. 63H)}$$

[**Hint :** Let $\lambda Pdx + \lambda Qdy + \lambda Rdz = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

This $\Rightarrow \frac{\partial u}{\partial x} = \lambda P, \frac{\partial u}{\partial y} = \lambda Q, \frac{\partial u}{\partial z} = \lambda R$.]

3.8. Total Differential Coefficient

Let $u = f(x, y)$

where $x = \phi(t)$ and $y = \psi(t)$.

Then from the previous article 3.6

$$\delta u = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \delta x$$

$$+ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \delta y$$

$$\frac{\delta u}{\delta t} = \frac{f(x+\delta x,\ y+\delta y) - f(x,\ y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t}$$

$$+ \frac{f(x,\ y+\delta y) - f(x,\ y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Now taking the limit as $\delta t \to 0$ and consequently as $\delta x \to 0$ and $\delta y \to 0$, then exactly as in the last article

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Note : Since x and y both are functions of a single variable t, therefore their differentials are written total, but since u is a function of two variables x and y, therefore its differential is written partial. We can express u in terms of a single variable t by substituting $x = \phi(t)$ and $y = \psi(t)$ in $u = f(x, y)$. Therefore the total differential $\frac{du}{dt}$ of u w.r. to t has been written.

Generalisation : Similarly if

$u = f(x, y, z)$ and $x = \phi_1(t)$, $y = \phi_2(t)$ and $z = \phi_3(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

This formula can be generalised similarly up to n variables $x_1, x_2, x_3, \ldots x_n$.

Cor. If y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

3.8. Change of variables

In Art. 3.7, we have learnt to determine $\frac{du}{dt}$ if $u = f(x, y)$ where x and y are functions of a single variable t.

Here if $u = f(x, y)$ be a function of two variables x and y when x, y themselves are functions of other two variables t_1, t_2 say $x = \phi(t_1, t_2)$, $y = \psi(t_1, t_2)$, then the formula of art. 3.8, will be of the following form:

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}.$$

In the above formulae, the values of $\frac{\partial u}{\partial t_1}$ and $\frac{\partial u}{\partial t_2}$ have been expressed in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

It has to be noted that if $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$, then we can express t_1, t_2 in terms of x and y; for example

$$t_1 = F_1(x, y), t_2 = F_2(x, y).$$

In that case we can evaluate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ according to the above formuale; e.g.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y}.$$

[See Ex. 3, 6, 8 worked out for their elucidation.]

Again if $u = f(x_1, x_2, x_3)$ and if x_1, x_2, x_3 are each known functions of variables x, y, z, we shall have similarly,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial z}.$$

This formula can be generalised to the case when $u = f(x_1, x_2, x_3, \ldots, x_n)$ and each x_i itself is a function of several variables $x, y, z, \ldots$

3.10. The second proof of Euler's Theorem

Let $f(x, y)$ be a homogeneous function of nth order. Therefore,

$$f(tx, ty) = t^n f(x, y). \qquad \ldots(1)$$

We shall assume here that in the equation (1), x and y are constants and t is a variable quantity so that both the sides of the equation (1) are functions of t. Therefore, we can differentiate both sides of (1) w.r.t. t.

First, we differentiate the L.H.S. w.r.t. t. For this, we put $tx = u$ and $ty = v$.

We know that

$$\frac{df(u, v)}{dt} = \frac{\partial f(u, v)}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f(u, v)}{\partial v} \cdot \frac{dv}{dt} \qquad ...(2)$$

Now from $tx = u$ we have $\frac{du}{dt} = x$ and from $ty = v$ we have, $\frac{dv}{dt} = y$.

$\therefore$ from (2),

$$\frac{d}{dt} f(u, v) = \frac{\partial f(u, v)}{\partial u} \cdot x + \frac{\partial f(u, v)}{\partial v} \cdot y$$

i.e. $$\frac{df(tx, ty)}{dt} = x\frac{\partial f(tx, ty)}{\partial (tx)} + y\frac{\partial f(tx, ty)}{\partial (ty)}. \qquad ...(3)$$

Now the differential of R.H.S. of (1) $= nt^{n-1} f(x, y)$ ...(4)

Therefore from (3) and (4),

$$x\frac{\partial f(tx, ty)}{\partial (tx)} + y\frac{\partial f(tx, ty)}{\partial (ty)} = nt^{n-1} f(x, y)$$

which is true for every value of x, y, t.

In particular, putting $t = 1$,

$$x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y} = nf(x, y).$$

Alternative proof :

Let u be a homogeneous function of the nth order in x, y and z so that we can write

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) \qquad ...(1)$$

Let $$\frac{y}{x} = Y \text{ and } \frac{z}{x} = Z.$$

Then $$\frac{\partial Y}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial Y}{\partial y} = \frac{1}{x}, \quad \frac{\partial Y}{\partial z} = 0.$$

and $$\frac{\partial Z}{\partial x} = -\frac{z}{x^2}, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial Z}{\partial z} = \frac{1}{x}.$$

Now, from (1), we have,

$$u = x^n f(Y, Z) \qquad ...(2)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f + x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial x} \right\}$$

$$= nx^{n-1} f + x^n \left\{ \frac{\partial f}{\partial Y} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial f}{\partial Z} \left(-\frac{z}{x^2} \right) \right\}$$

$$= nx^{n-1} f - x^{n-2} \left\{ y \frac{\partial f}{\partial Y} + z \frac{\partial f}{\partial Z} \right\}. \qquad \ldots(3)$$

Again, differentiating (2) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial y} \right\}$$

$$= x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{1}{x} + \frac{\partial f}{\partial Z} \cdot 0 \right\}$$

$$= x^{n-1} \frac{\partial f}{\partial Y}. \qquad \ldots(4)$$

Similarly $$\frac{\partial u}{\partial Z} = x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial z} \right\}$$

$$= x^n \left\{ \frac{\partial f}{\partial Y} \cdot 0 + \frac{\partial f}{\partial Z} \cdot \frac{1}{x} \right\}$$

$$= x^{n-1} \cdot \frac{\partial f}{\partial Z}. \qquad \ldots(5)$$

Hence from (3), (4) and (5), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= nx^n f - x^{n-1} \left\{ y \frac{\partial f}{\partial Y} + z \frac{\partial f}{\partial Z} \right\} + x^{n-1} y \frac{\partial f}{\partial Y} + x^{n-1} z \frac{\partial f}{\partial Z}$$

$$= nx^n f = nu.$$

Note : This method is quite general and can be applied whatever be the number of variables.

3.11. First differential coefficient of an implict function

Let $f(x, y) = 0$.

Then from the previous article, we know that

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \text{ if } y \text{ is a function of } x.$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f + x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial x} \right\}$$

$$= nx^{n-1} f + x^n \left\{ \frac{\partial f}{\partial Y} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial f}{\partial Z} \left(-\frac{z}{x^2} \right) \right\}$$

$$= nx^{n-1} f - x^{n-2} \left\{ y \frac{\partial f}{\partial Y} + z \frac{\partial f}{\partial Z} \right\}. \qquad \ldots(3)$$

Again, differentiating (2) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial y} \right\}$$

$$= x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{1}{x} + \frac{\partial f}{\partial Z} \cdot 0 \right\}$$

$$= x^{n-1} \frac{\partial f}{\partial Y}. \qquad \ldots(4)$$

Similarly $$\frac{\partial u}{\partial Z} = x^n \left\{ \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial z} \right\}$$

$$= x^n \left\{ \frac{\partial f}{\partial Y} \cdot 0 + \frac{\partial f}{\partial Z} \cdot \frac{1}{x} \right\}$$

$$= x^{n-1} \cdot \frac{\partial f}{\partial Z}. \qquad \ldots(5)$$

Hence from (3), (4) and (5), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= nx^n f - x^{n-1} \left\{ y \frac{\partial f}{\partial Y} + z \frac{\partial f}{\partial Z} \right\} + x^{n-1} y \frac{\partial f}{\partial Y} + x^{n-1} z \frac{\partial f}{\partial Z}$$

$$= nx^n f = nu.$$

Note : This method is quite general and can be applied whatever be the number of variables.

3.11. First differential coefficient of an implict function

Let $f(x, y) = 0$.

Then from the previous article, we know that

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \text{ if } y \text{ is a function of } x.$$

$$\Rightarrow \qquad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}, \text{ since } f(x, y) = 0 = u.$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

This is a very useful formula for finding out $\frac{dy}{dx}$ when the relationship between x and y is given in the form of an implicit function.

3.12. Second differential coefficient of an implicit function

Let $f(x, y) = 0$.

Then from the last article,

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}. \qquad \ldots(1)$$

Now we denote $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$ respectively by the symbols p, q, r, s.

Then from (1), $\frac{dy}{dx} = -\frac{p}{q}$.

Therefore $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{p}{q}\right) = -\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}$. $\qquad \ldots(2)$

But $\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dx} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx}$$

$$= r + s\left(-\frac{p}{q}\right) = \frac{qr - ps}{q}.$$

Similarly $\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dx}$

$$= \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dx} = s + t\left(-\frac{p}{q}\right) = \frac{qs - pt}{q}.$$

Hence putting the value of $\frac{dp}{dx}$, $\frac{dq}{dx}$ in (2), we get

$$\frac{d^2y}{dx^2} = -\frac{q\frac{qr - ps}{q} - p\frac{qs - pt}{q}}{q^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

3.13. Taylor's Theorem for a function of two variables.

As in the case of Taylor's Theorem for a function of one variable, we have (keeping y constant),

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k) + \ldots\ldots \quad \ldots(1)$$

But $f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \ldots.$

Hence, from (1),

$$f(x+h, y+k) = f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \ldots$$
$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \ldots\right\}$$
$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \ldots.\right\} + \ldots.$$
$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$
$$+ \left\{\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + hk\frac{\partial^2 f(x, y)}{\partial x \partial y} + \frac{h^2}{2!}\frac{\partial^2 f(x, y)}{\partial x^2}\right\} + \ldots$$
$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$
$$+ \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\} + \ldots.$$

which can be symbolically written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \ldots$$

Worked out Examples

EXAMPLE 1. *If* $ax^2 + by^2 + cz^2 = 1$ *and* $lx + my + nz = 0$, *prove that*

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}.$$

SOLUTION : Since $ax^2 + by^2 + cz^2 = 1$, therefore from the formula

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0$$

we get $a.2xdx + b.2ydy + c.2zdz = 0$

$\Rightarrow$ $axdx + bydy + czdz = 0.$ ...(1)

Again since $lx + my + nz = 0$, therefore

$ldx + mdy + ndz = 0.$ ...(2)

Now solving (1) and (2) by cross-multiplication, we have

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}.$$

EXAMPLE 2. *If* $x^2 + y^2 + z^2 - 2xyz = 1$, *prove that*

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

SOLUTION : The given function is

$$x^2 - 2xyz = 1 - y^2 - z^2 \Rightarrow x^2 - 2xy + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$\Rightarrow$ $(x - yz)^2 = (1 - y^2)(1 - z^2)$

$\Rightarrow$ $(1 - x^2)(x - yz)^2 = (1 - x^2)(1 - y^2)(1 - z^2) = k^2$ (say)

$\therefore$ $(x - yz)^2 = \dfrac{k^2}{1 - x^2} \Rightarrow x - yz = \dfrac{k}{\sqrt{1 - x^2}}$

Similarly $y - zx = \dfrac{k}{\sqrt{1 - y^2}}$ and $z - xy = \dfrac{k}{\sqrt{1 - z^2}}$ ...(1)

Now, differentiating the given function, we get

$$2xdx + 2ydy + 2zdz - 2yzdx - 2zxdy - 2xydz = 0$$

$\Rightarrow$ $(x - yz)dx + (y - zx)dy + (z - xy)dz = 0$

$\Rightarrow$ $\dfrac{kdx}{\sqrt{1 - x^2}} + \dfrac{kdy}{\sqrt{1 - y^2}} + \dfrac{kdz}{\sqrt{1 - z^2}} = 0.$

Hence the result.

EXAMPLE 3. *If* $f(x, y) = 0$, $\phi(y, z) = 0$, *show that*

$$\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

SOLUTION : Given that $f(x, y) = 0$

$\therefore$ $\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x} \Big/ \dfrac{\partial f}{\partial y}.$ ...(1)

Again, since $\phi(y, z) = 0$

$\therefore$ $\dfrac{dy}{dz} = -\dfrac{\partial \phi}{\partial z} \Big/ \dfrac{\partial \phi}{\partial y}.$ ...(2)

Now dividing (1) by (2), we get

$$\frac{dy}{dx}\cdot\frac{dz}{dy}=\frac{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial z}}$$

$$\therefore \quad \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}.$$

EXAMPLE 4. *If* $u = f(y - z, z - x, x - y)$, *prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$ **(P.U. 58H, B.U. 62H, 87; M.U. 71H)**

SOLUTION : Let $X = y - z$, $Y = z - x$, $Z = x - y$.

Then $u = f(X, Y, Z)$.

Now $$\frac{\partial X}{\partial x}=0,\ \frac{\partial X}{\partial y}=1,\ \frac{\partial X}{\partial z}=-1$$

$$\frac{\partial Y}{\partial x}=-1,\ \frac{\partial Y}{\partial y}=0,\ \frac{\partial Y}{\partial z}=1$$

$$\frac{\partial Z}{\partial x}=1,\ \frac{\partial Z}{\partial y}=-1,\ \frac{\partial Z}{\partial z}=0.$$

Now $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$=0+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)$$

$$=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}. \qquad ...(1)$$

Similarly $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \qquad ...(2)$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial Z}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$$=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}. \qquad ...(3)$$

In the end, adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

EXAMPLE 5. *If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that*

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

(P.U. 35H, 43H, 49H, 53H, 60H; B.U. 56H, 60H)

SOLUTION : We know from properties of triangle, that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Given that R is constant. Therefore.

$$\frac{a}{\sin A} = 2R = k \quad \text{(say)} \quad \therefore \quad a = k\sin A,$$

Hence $da = k\cos A\, dA.$

Similarly $db = k\cos B\, dB,$

$dc = k\cos C\, dC.$

$$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = k(dA + dB + dC) = k \times 0 = 0.$$

since $A + B + C = \pi \quad \therefore \quad dA + dB + dC = 0.$

EXAMPLE 6. *If the three sides a, b, c of a triangle ABC are measured, prove that the error in the angle, A, due to small errors in the sides, is given by*

$$\delta A = \frac{\sin A}{\sin B \sin C} \cdot \frac{\delta a}{a} - \cot C \cdot \frac{\delta b}{b} - \cot B \cdot \frac{\delta c}{c}.$$

(P.U. 50H. B.U. 54H)

SOLUTION : We have,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \quad b^2 + c^2 - a^2 = 2bc\cos A$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A.$$

Differentiating, we get

$$2ada = 2bdb + 2cdc - 2(ccosAdb + bcosAdc - bcsinAdA)$$

$$\Rightarrow \quad ada = bdb + cdc - ccosAdb - bcosAdc + bcsinAdA$$

$$= (b - c\cos A)db + (c - b\cos A)dc + bc\sin AdA$$

$$= a\cos Cdb + a\cos Bdc + bc\sin AdA,$$

since $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

$$= a\cos Cdb + a\cos Bdc + ca\sin Bda,$$

since $\frac{a}{\sin A} = \frac{b}{\sin B} \quad \therefore \quad b\sin A = a\sin B$

$$\Rightarrow \quad da = \cos C db + \cos B dc + c\sin B dA$$

$$\Rightarrow \quad c\sin B dA = da - \cos C db - \cos B dc$$

$$\Rightarrow \quad dA = \frac{1}{c\sin B} da - \frac{\cos C}{c\sin B} db - \frac{\cos B}{c\sin B} dc$$

$$= \frac{a}{c\sin B}\frac{da}{a} - \frac{b\cos C}{c\sin B}\frac{db}{b} - \frac{\cos B}{\sin B}\frac{dc}{c}$$

$$= \frac{\sin A}{\sin B \sin C}\frac{da}{a} - \cot C \frac{db}{b} - \cot B \frac{dc}{c}.$$

EXAMPLE 7. *If* $u = f(x, y)$, $x = r\cos\theta$ *and* $y = r\sin\theta$, *prove that*

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

SOLUTION : Since $u = f(x, y)$ where x and y are functions of r and θ, we get

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} \qquad ...(1)$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}. \qquad ...(2)$$

But $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

and

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Hence from (1) and (2), we get

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad ...(3)$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta). \qquad ...(4)$$

Now squaring (3) and (4), we get the L.H.S.

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\cdot r^2\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)^2.$$

$$= \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Hence the result.

EXAMPLE 8. *If z and u be functions of x and y defined by $\{z - \phi(u)\}^2 = x^2(y^2 - u^2)$ and $\{z - \phi(u)\}\phi'(u) = ux^2$, prove that*

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = xy.$$ **(P.U. 41H, 62H. Bhag. 67H)**

SOLUTION : Since z is a function of x and y, therefore

$$dz = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}dy. \quad ...(1)$$

Given : $\{z - \phi(u)\}^2 = x^2(y^2 - u^2)$ ...(2)

and $\{z - \phi(u)\}\phi'(u) = ux^2.$ ...(3)

Now, taking the differential of (2), we get

$$2\{z - \phi(u)\}\{dz - \phi'(u)du\}$$

$$= x^2(2ydy - 2udu) + (y^2 - u^2)2xdx$$

$$\Rightarrow \quad \{z - \phi(u)\}dz - \{z - \phi(u)\}\phi'(u)du$$

$$= x^2ydy - ux^2du + xy^2dx - u^2xdx$$

$$\Rightarrow \quad \{z - \phi(u)\}dz - ux^2dx$$

$$= x^2ydy - ux^2 + xy^2dx - u^2xdx; \text{ because of (3)}$$

$$\Rightarrow \quad \{z - \phi(u)\}dz = x^2ydy + x(y^2 - u^2)dx. \quad ...(4)$$

Now, the L.H.S. is

$$= \{z - \phi(u)\}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right); \text{ because of (1)}$$

$$= \{z - \phi(u)\}\frac{\partial z}{\partial x}dx + \{z - \phi(u)\}\frac{\partial z}{\partial y}dy.$$

Hence from (4), we have

$$\{z - \phi(u)\}\frac{\partial z}{\partial x}dx + \{z - \phi(u)\}\frac{\partial z}{\partial y}dy$$

$$= x(y^2 - u^2)dx + x^2ydy.$$

Thus equating the coeff. of dx and dy from both sides, we get

$$\{z - \phi(u)\}\frac{\partial z}{\partial x} = x(y^2 - u^2) \quad ...(5)$$

and $$\{z - \phi(u)\}\frac{\partial z}{\partial y} = x^2y. \quad ...(6)$$

Now, multiplying (5) and (6), we get

$$\{z - \phi(u)\}^2 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = x^3y(y^2 - u^2)$$

$$\Rightarrow \qquad x^2(y^2 - u^2)\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = x^3 y(y^2 - u^2); \text{ from (2)}$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = xy.$$

Hence proved.

EXAMPLE 9. *If u be a homogeneous function of nth degree in x, y, z and if $u = f(X, Y, Z)$ where X, Y, Z are the first differential coefficients of u with respect to x, y, z respectively, prove that*

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \frac{n}{n-1}u.$$

(P.U. 55H, 59H, 65H; B.U. 62H; R.U. 64H; Bhag. 90H, 94H)

SOLUTION : Since u is a homogeneous function of degree n in three variables x, y and z, therefore by Euler's Theorem we have,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu. \qquad ...(1)$$

Also, $u = f(X, Y, Z)$

where $X = \dfrac{\partial u}{\partial x}, \quad Y = \dfrac{\partial u}{\partial y}, \quad Z = \dfrac{\partial u}{\partial z}.$ ...(2)

Now, we have

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial x} \qquad ...(3)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial y} \qquad ...(4)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial z}. \qquad ...(5)$$

Multiplying (3), (4) and (5) by x, y, z respectively and adding, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\partial f}{\partial X}\left\{x\frac{\partial X}{\partial x} + y\frac{\partial X}{\partial y} + z\frac{\partial X}{\partial z}\right\}$$

$$+ \frac{\partial f}{\partial Y}\left\{x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z}\right\} + \frac{\partial f}{\partial Z}\left\{x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} + z\frac{\partial Z}{\partial z}\right\}$$

which because of (2) becomes

$$= \frac{\partial f}{\partial X}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right\}$$

$$+ \frac{\partial f}{\partial Y}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right\}$$

$$+ \frac{\partial f}{\partial Z}\left\{x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right\}$$

$$\Rightarrow \quad nu = \frac{\partial f}{\partial X}\left\{x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x} + z\frac{\partial^2 u}{\partial z \partial x}\right\} + \frac{\partial f}{\partial Y}\left\{x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial z \partial y}\right\}$$

$$+ \frac{\partial f}{\partial Z}\left\{x\frac{\partial^2 u}{\partial x \partial z} + y\frac{\partial^2 u}{\partial y \partial z} + z\frac{\partial^2 u}{\partial z^2}\right\} \quad ...(6)$$

Now, differentiating (1) partially w.r.t. x, (keeping y and z const.), we get

$$\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + \left(y\frac{\partial^2 u}{\partial x \partial y}\right) + \left(z\frac{\partial^2 u}{\partial x \partial z}\right) = n\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = (n-1)\frac{\partial u}{\partial x} = (n-1)X. \quad ...(7)$$

Similarly differentiating (1) partially w.r.t. y and z, seperately, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial y \partial z} = (n-1)Y \quad ...(8)$$

and

$$x\frac{\partial^2 u}{\partial z \partial x} + y\frac{\partial^2 u}{\partial z \partial y} + z\frac{\partial^2 u}{\partial z^2} = (n-1)Z \quad ...(9)$$

Hence from (7), (8) and (9), the equation (6) becomes

$$nu = \frac{\partial f}{\partial X}\{(n-1)X\} + \frac{\partial f}{\partial Y}\{(n-1)Y\} + \frac{\partial f}{\partial Z}\{(n-1)Z\}$$

$$= (n-1)\left\{X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z}\right\}$$

$$\therefore \quad X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \frac{n}{n-1}u.$$

Example 11

1. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$.

2. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^3$, find the value of $\frac{dz}{dx}$ when $x = a$, $y = a$.

3. If the curves $f(x, y) = 0$, $\phi(x, y) = 0$ touch, show that at the point of contact

$$\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x} = 0.$$

Hint : Two curves touch each other at a point where the values of $\frac{dy}{dx}$ for the two curves are the same.]

$\frac{dy}{dx}$

4. If $z = xyf\left(\frac{y}{x}\right)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$ and if z is a constant, then

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}.$$

[For the second part if z is a constant, then $xyf\left(\frac{y}{x}\right) = \text{const.}$

Hence $\frac{\partial}{\partial x}\left\{xyf\left(\frac{y}{x}\right)\right\} + \frac{\partial}{\partial y}\left\{xyf\left(\frac{y}{x}\right)\right\}\frac{dy}{dx} = 0$ etc.]

5. If z is a function of x and y and if $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

[Use $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ etc.]

6. If u and v are functions of x and y defined by $x = u + e^{-v}\sin u$, $y = u + e^{-v}\cos u$, prove that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. **(P.U. 36S)**

7. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that $\frac{d^2y}{dx^2} = -\frac{a}{(1+x^2)^{3/2}}$. **(B.U. 85)**

8. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$, prove that

$$\frac{dx}{\frac{ny}{b^2} - \frac{mz}{c^2}} = \frac{dy}{\frac{lz}{c^2} - \frac{nx}{a^2}} = \frac{dz}{\frac{mx}{a^2} - \frac{ly}{b^2}}.$$

9. If A, B, C be the angles of a triangle such that $\sin^2A + \sin^2B + \sin^2C = \text{const.}$, prove that

$$\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}.$$ **(P.U. 63A; Bhag. 63A, 65S)**

[**Hint :** $\because\ A + B + C = \pi$

$$\therefore\ \frac{dA}{dB} + 1 + \frac{dC}{dB} = 0. \qquad \ldots(1)$$

Again, let $f \equiv \sin^2A + \sin^2B + \sin^2C = \text{const.}$

$\therefore \quad \frac{\partial f}{\partial A} = 2\sin A \cos A = \sin 2A$ etc.

Now $\frac{\partial f}{\partial A} \cdot \frac{dA}{dB} + \frac{\partial f}{\partial B} \cdot \frac{dB}{dB} + \frac{\partial f}{\partial C} \cdot \frac{dC}{dB} = 0$

$\Rightarrow \quad \sin 2A \frac{dA}{dB} + \sin 2B + \sin 2C \frac{dC}{dB} = 0$ ...(2)

We have now to eliminate $\frac{dC}{dB}$ from (1) and (2).]

10. The side c of a triangle is determined by measuring the sides a, b and the angle C. If the possible errors of measurement are da, db and dC respectively, show that

(i) $dc = \cos B da + \cos A db + a \sin B dC$ **(P.U. 58H; R.U. 66H. M.U. 73H)**

(ii) If c is constant, then $\cos B.da + \cos A.dB = 0$.

[**Hint :** We have $c^2 = a^2 + b^2 - 2ab \cos C$

$\therefore \quad 2cdc = 2ada + 2bdb - 2(b\cos C da + a\cos C db - ab\sin C dC)$

$\Rightarrow \quad cdc = ada + bdb - b\cos C da - a\cos C db + ad\sin C dC$

$= (a - b\cos C)da + (b - a\cos C)db + ab\sin C dC$

$= \{(b\cos C + c\cos B) - b\cos C\}da + \{(c\cos A + a\cos C)$

$- a\cos C\}db + ab\sin C dC$

$= c\cos B da + c\cos A db + ab\sin C dC$

$= c\cos B da + c\cos A db + ac\sin B dC$; since $\frac{b}{c} = \frac{\sin B}{\sin C}$

$\Rightarrow \quad dc = \cos B da + \cos A db + a\sin B dC.$]

11. If Δ be the area of the ΔABC, prove that

$$d\Delta = R(\cos A da + \cos B db + \cos C dc)$$

where R is the circum-radius of the ΔABC. **(M.U. 86)**

12. If $x = r\cos\theta$, $y = r\sin\theta$, transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates. **(Kanpur 86**

[Here $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$.]

13. If z be a function of x, y and u, v be two other variables such that $u = lx + my$, $v = ly - mx$, show that

(i) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = (l^2 + m^2)\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}$

(ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

ANSWERS

1. $2xy - \frac{x^2(2x + y)}{x + 2y}$ **2.** 0 **12.** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}$.

4

INDETERMINATE FORMS

4.1. We know that if we put $x = 0$ in the expression $\frac{\sin x}{x}$, then it reduces to the form $\frac{o}{o}$ which has got no meaning. Thus if any expression $\frac{f(x)}{\phi(x)}$ becomes of the form $\frac{0}{0}$ at the point $x = a$, then this form is called indeterminate i.e. the value of $\frac{f(x)}{\phi(x)}$ at that point is indefinite. This indeterminacy is due to the fact that 0 is not any finite and definite number in the sense in which we take 2, 3, 4, ... 10. 0 means a very very small positive number. In this sense $\frac{1}{10^{60}}$ is a small number and in the same sense $\frac{1}{10^{100}}$ or $\frac{1}{10^{50}}$ is also a small number . Hence if we represent 0 by $\frac{1}{10^{60}}$ or $\frac{1}{10^{100}}$ or $\frac{1}{10^{50}}$, then $\frac{0}{0}$ will be different in different situations. That is, $\frac{0}{0}$ is an indefinite quantity. In the same manner, we take ∞ to be a very very large quantity. If the value of any expression $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ at the point $x = a$, then this form is called indeterminate. Thus $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are two indeterminate forms. Apart from these, the other types of indeterminate forms are:

$$0 \times \infty,\ \infty - \infty,\ 0^0,\ 1^\infty \text{ and } \infty^0.$$

Of all these forms, $\frac{0}{0}$ is considered to be the fundamental indeterminate form, for other types of indeterminate forms can be reduced to the form $\frac{0}{0}$. The value of the indeterminate form $\frac{0}{0}$ is calculated by using L'Hospital Rule which will be explained in the next article.

De L' Hospital (1661–1704) was a French mathematician.

Note : While evaluating limit, it is frequently convenient to use the expansion of some important functions; for example,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \text{ etc.}$$

We shall use the expansion of the given function in the form of a series or L' Hospital rule whichever is convenient.

4.2. I. Form $\frac{0}{0}$: L' Hospital's Rule

Let $\phi(x)$ and $\psi(x)$ be functions of x capable of being expanded by Taylor's theorem and if

$\phi(a) = 0$ *and* $\psi(a) = 0$, *then*

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}$$

provided the latter limit exists (finite or infinite).

Proof. Let $x = a + h$, then as $x \to a$, $h \to 0$.

Now, according to Taylor's Theorem,

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \ldots$$

$$\psi(a+h) = \psi(a) + h\psi'(a) + \frac{h^2}{2!}\psi''(a) + \ldots$$

$$\therefore \quad \frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \ldots}{\psi(a) + h\psi'(a) + \frac{h^2}{2!}\psi''(a) + \ldots}$$

$$= \frac{h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \ldots}{h\psi'(a) + \frac{h^2}{2!}\psi''(a) + \ldots}$$

since $\phi(a) = 0$; $\psi(a) = 0$.

$$= \frac{\phi'(a) + \frac{h}{2!}\phi''(a) + \ldots}{\psi'(a) + \frac{h}{2!}\psi''(a) + \ldots} \qquad (1)$$

dividing by h.

Now taking the limit when $h \to 0$,

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{h \to 0}{Lt} \frac{\phi(a+h)}{\psi(a+h)} = \frac{\phi'(a)}{\psi'(a)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

Again if $\frac{\phi'(a)}{\psi'(a)} = \frac{0}{0}$, then proceeding as before

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \frac{\phi''(a)}{\psi''(a)} = \underset{x \to a}{Lt} \frac{\phi''(x)}{\psi''(x)}.$$

Generally if $\phi'(a) = 0$, $\phi''(a) = 0 \ldots \phi^{n-1}(a) = 0$,

and $\psi'(a) = 0 = \psi''(a) = \ldots = \psi^{n-1}(a)$, then

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \frac{\phi^n(a)}{\psi^n(a)} = \underset{x \to a}{Lt} \frac{\phi^n(x)}{\psi^n(x)}.$$

In other words, under this rule, we shall go on differentiating successively the numerator $\phi(x)$ and the denominator $\psi(x)$ separately and each time we shall put $x = a$ in the differential coeff. This process will continue till the form $\frac{0}{0}$ stops to occur.

EXAMPLE 1. *Limit of* $\frac{\sin\theta}{\theta}$ *when* $\theta \to 0$.

SOLUTION: This is of the form $\left[\frac{0}{0}\right]$.

$$\therefore \text{ The given limit} = \underset{\theta \to 0}{Lt} \frac{\text{d.c. of } \sin\theta}{\text{d.c. of } \theta}$$

$$= \underset{\theta \to 0}{Lt} \frac{\cos\theta}{1} = \cos 0 = 1.$$

EXAMPLE 2. *Limit of* $\frac{\tan\theta}{\theta}$ *when* $\theta \to 0$.

SOLUTION: This is of the form $\left[\frac{0}{0}\right]$.

$$\therefore \text{ The given limit} = \underset{\theta \to 0}{Lt} \frac{\text{d.c. of } \tan\theta}{\text{d.c. of } \theta}$$

$$= \underset{\theta \to 0}{Lt} \frac{\sec^2\theta}{1} = \sec^2 0 = 1.$$

Worked out Examples

EXAMPLE 1. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x}$. **(R.U. 62A)**

SOLUTION : We know that $a^x = e^{x \log a}$.

$\therefore$ The given expression $= \dfrac{e^{x \log a} - 1}{x}$ $\left[\text{Form } \dfrac{0}{0}\right]$

$$\therefore \underset{x \to 0}{Lt} \frac{a^x - 1}{x} = \underset{x \to 0}{Lt} \frac{e^{x \log a} - 1}{x}$$

$$= \underset{x \to 0}{Lt} \frac{\text{d.c. of } (e^{x \log a} - 1)}{\text{d.c. of } x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{x \log a} \times \log a - 0}{1}$$

$$= e^0 \times \log a = \log a.$$

EXAMPLE 2. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log(1 - x^2)}{\log \cos x}$.

SOLUTION : This is of the form $\dfrac{0}{0}$.

$$\therefore \text{The given limit} = \underset{x \to 0}{Lt} \frac{\dfrac{1}{1 - x^2} \times (-2x)}{\dfrac{1}{\cos x} \times (-\sin x)}$$

$$= \underset{x \to 0}{Lt} \frac{2x}{(1 - x^2)\tan x}$$

$$= 2 \underset{x \to 0}{Lt} \frac{1}{1 - x^2} \times \underset{x \to 0}{Lt} \left(\frac{x}{\tan x}\right)$$

$$= 2 \times 1 \times 1 = 2.$$

EXAMPLE 3. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log(1 + kx^2)}{1 - \cos x}$. **(M.U. 68A)**

SOLUTION : This is of the form $\dfrac{0}{0}$.

$$\therefore \text{The given limit} = \underset{x \to 0}{Lt} \frac{\dfrac{1}{1 + kx^2} \times k \times 2x}{0 + \sin x}$$

$$= \underset{x \to 0}{Lt} \frac{2kx}{(1 + kx^2)\sin x}$$

$$= 2k \underset{x \to 0}{Lt} \frac{1}{1 + kx^2} \times \underset{x \to 0}{Lt} \left(\frac{x}{\sin x}\right) = 2k \times 1 \times 1 = 2k.$$

EXAMPLE 4. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$.

SOLUTION : This is of the form $\dfrac{0}{0}$.

$$\therefore \text{ The given limit } = \underset{x \to 0}{Lt} \frac{5\cos x - 7 \times 2\cos 2x + 3 \times 3\cos 3x}{\sec^2 x - 1}$$

$$= \underset{x \to 0}{Lt} \frac{5\cos x - 14\cos 2x + 9\cos 3x}{\sec^2 x - 1} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{-5\sin x + 28\sin 2x - 27\sin 3x}{2\sec x \sec x \tan x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{-5\cos x + 56\cos 2x - 81\cos 3x}{2[\sec^2 x.\sec^2 x + \tan x.2\sec^2 x.\tan x]}$$

$$= \frac{-5 + 56 - 81}{2[1 + 0]} = \frac{-30}{2} = -15.$$

EXAMPLE 5. *Prove that* $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1 + x)}{x^2} = \dfrac{3}{2}$.

(Bhag. 63S; B.U. 61A; P.U. 63A; M.U. 66S, 91; R.U. 73A)

SOLUTION : Putting $x = 0$, the given expression $= \dfrac{0}{0}$.

We find out the limit of this expression by using the expansion of e^x and $\log(1 + x)$.

The given expression

$$= \frac{x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)}{x^2}$$

$$= \frac{\left(x^2 + \frac{x^2}{2}\right) + x^3\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots}{x^2}$$

$$= \frac{3}{2} + \frac{1}{6}x + \text{higher powers of } x.$$

$\therefore$ The reqd. limit $= \dfrac{3}{2}$ when $x \to 0$.

EXAMPLE 6. *Find the values of a and b, so that*

$$\underset{x \to 0}{Lt} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1. \qquad \textbf{(Kanpur 87)}$$

SOLUTION : The given expression

$$= \frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^3}$$

$$= \frac{(1+a-b)x+\left(\frac{b}{6}-\frac{a}{2}\right)x^3+\ldots}{x^3}$$

If the limit of this expression is 1 when $x \to 0$, then

$$1+a-b=0 \text{ and } \frac{b}{6}-\frac{a}{2}=1$$

i.e. $a-b=-1$ and $b-3a=6$.

Adding these equations, we get

$$-2a=5 \quad \therefore \quad a=-\frac{5}{2}$$

$$\therefore \quad b=a+1=1-\frac{5}{2}=-\frac{3}{2}.$$

Hence $a=-\frac{5}{2}$ and $b=-\frac{3}{2}$.

EXAMPLE 7. *Evaluate* $\underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}}{x-\sin x}$

(P.U. 33S, 62A, 70S; R.U. 70S; B.U. 93H)

SOLUTION : It is of the form $\frac{0}{0}$.

$$\text{The reqd. limit} = \underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}}{x-\sin x}$$

$$= \underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}\cos x}{1-\cos x} \qquad [\text{Form } \tfrac{0}{0}]$$

$$= \underset{x \to 0}{Lt} \frac{e^x-\{e^{\sin x}(-\sin x)+\cos^2 x.e^{\sin x}\}}{\sin x} \qquad [\text{Form } \tfrac{0}{0}]$$

$$= \underset{x \to 0}{Lt} \frac{[e^x+\{e^{\sin x}\cos x+\sin x e^{\sin x}.\cos x\}-\{\cos^3 x e^{\sin x}-2\cos x \sin x e^{\sin x}\}]}{\cos x}$$

$$= \frac{1+(1)-(1)}{1}=1.$$

Alternative Method :

$$e^x-e^{\sin x}=e^x-\left[1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots.\right]$$

SOLUTION : The given expression

$$= \frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^3}$$

$$= \frac{(1+a-b)x+\left(\frac{b}{6}-\frac{a}{2}\right)x^3+\ldots}{x^3}$$

If the limit of this expression is 1 when $x \to 0$, then

$$1+a-b=0 \text{ and } \frac{b}{6}-\frac{a}{2}=1$$

i.e. $a-b=-1$ and $b-3a=6$.

Adding these equations, we get

$$-2a=5 \quad \therefore \quad a=-\frac{5}{2}$$

$$\therefore \quad b=a+1=1-\frac{5}{2}=-\frac{3}{2}.$$

Hence $a=-\frac{5}{2}$ and $b=-\frac{3}{2}$.

EXAMPLE 7. *Evaluate* $\underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}}{x-\sin x}$

(P.U. 33S, 62A, 70S; R.U. 70S; B.U. 93H)

SOLUTION : It is of the form $\frac{0}{0}$.

The reqd. limit $= \underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}}{x-\sin x}$

$$= \underset{x \to 0}{Lt} \frac{e^x-e^{\sin x}\cos x}{1-\cos x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{e^x-\{e^{\sin x}(-\sin x)+\cos^2 x.e^{\sin x}\}}{\sin x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{[e^x+\{e^{\sin x}\cos x+\sin x e^{\sin x}.\cos x\}-\{\cos^3 x e^{\sin x}-2\cos x \sin x e^{\sin x}\}]}{\cos x}$$

$$= \frac{1+(1)-(1)}{1}=1.$$

Alternative Method :

$$e^x-e^{\sin x}=e^x-\left[1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots.\right]$$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$$

$$- \left[1 + \left\{x - \frac{x^3}{3!} + \ldots\right\} + \frac{1}{2!}\left\{x - \frac{x^3}{3!} + \ldots\right\}^2 + \frac{1}{3!}\left\{x - \frac{x^3}{3!} + \ldots\right\}^3 + \ldots\right]$$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$$

$$- \left[1 + x + \frac{x^2}{2!} + x^3\left\{-\frac{1}{3!} + \frac{1}{3!}\right\} + x^4\left\{\frac{1}{2}\left(-\frac{2}{3!}\right) + \frac{1}{4!}\right\} + \ldots\right]$$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right] - \left[1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \ldots\right]$$

$$= \frac{x^3}{3!} + x^4 + \left\{\frac{1}{4!} + \frac{1}{8}\right\} + \ldots$$

$$= \frac{x^3}{6} + \frac{x^4}{6} + \ldots$$

and $x - \sin x = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

$$= \frac{x^3}{6} - \frac{x^5}{120} + ..$$

$$\therefore \quad \frac{e^x - e^{\sin x}}{x - \sin x} = \frac{\frac{x^3}{6} + \frac{x^4}{6} + \ldots}{\frac{x^3}{6} - \frac{x^5}{120} + \ldots} = \frac{\frac{1}{6} + \frac{x}{6} + \ldots}{\frac{1}{6} - \frac{x^2}{120} + \ldots}$$

$\therefore$ The reqd. limit when $x = 0$ is $\frac{1/6}{1/6} = 1$.

EXAMPLE 8. *Evaluate* $\underset{x \to \infty}{Lt}\, x\left[\left(1 + \frac{a}{x}\right)^x - e^a\right]$. **(B.U. 92H)**

SOLUTION : This is of the form $\infty \times 0$.

Put $x = 1/y$ so that when $x \to \infty$, $y \to 0$.

Therefore the given limit

$$= \underset{y \to 0}{Lt}\, \frac{1}{y}\left[(1 + ay)^{1/y} - e^a\right] \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{y \to 0}{Lt}\, \frac{\text{d.c. of } (1+ ay)^{1/y}}{1} \ldots(1); \text{ by using L' Hospital's rule.}$$

To evaluate (1), let $u = (1 + ay)^{1/y}$

$$\Rightarrow \qquad \log u = \frac{1}{y}\log(1 + ay)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dy} = \frac{1}{y}\cdot\frac{a}{1+ay} - \frac{\log(1+ay)}{y^2}$$

$$\Rightarrow \quad \frac{du}{dy} = (1+ay)^{1/y}\left[\frac{a}{y(1+ay)} - \frac{1}{y^2}\log(1+ay)\right] \qquad ...(2)$$

But $$\frac{a}{y(1+ay)} = \frac{a}{y}(1+ay)^{-1} = \frac{a}{y}\{1 - ay + a^2y^2 - \ldots\}$$

$$= \frac{a}{y} - a^2 + a^3y - a^4y^2 + \ldots;$$

and $$\frac{1}{y^2}\log(1+ay) = \frac{1}{y^2}\left\{ay - \frac{a^2y^2}{2} + \frac{a^3y^3}{3} - \ldots\right\}$$

$$= \frac{a}{y} - \frac{a^2}{2} + \frac{a^3y}{3} - \ldots$$

On subtraction, $$\frac{a}{y(1+ay)} - \frac{1}{y^2}\log(1+ay)$$

$$= \left(-a^2 + \frac{a^2}{2}\right) + y\left(a^3 - \frac{a^3}{3}\right) + ..$$

$$= -\frac{a^2}{2} + \frac{2a^3}{3}y + \text{higher powers of } y.$$

$\therefore$ From (2), $$\frac{du}{dy} = (1+ay)^{1/y}\left\{-\frac{a^2}{2} + \frac{2a^3}{3}y + \text{higher powers of } y\right\}$$

Hence from (1) the required limit

$$= \lim_{y\to 0}(1+ay)^{1/y}\left\{-\frac{a^2}{2} + \frac{2}{3}a^3y + \text{higher powers of } y\right\}$$

$$= e^a \times \left(-\frac{a^2}{2}\right) = -\frac{a^2}{2}e^a.$$

EXAMPLE 9. *Evaluate* $\lim_{x\to 0}\dfrac{(1+x)^{1/x} - e}{x}$.

(B.U. 62S, 73A; P.U. 57S, 61A, 64A, 95H; M.U. 71A, 75A, 81A, 87, 90; Kanpur 86)

SOLUTION : When $x = 0$ the given expression $= \dfrac{e-e}{0} = \dfrac{0}{0}$.

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0}\frac{\text{d.c. of }\{(1+x)^{1/x} - e\}}{1};$$

by using L' Hospital's rule

$$= \lim_{x\to 0}\{\text{d.c. of } (1+x)^{1/x}\} \qquad ...(1)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dy} = \frac{1}{y}\cdot\frac{a}{1+ay} - \frac{\log(1+ay)}{y^2}$$

$$\Rightarrow \quad \frac{du}{dy} = (1+ay)^{1/y}\left[\frac{a}{y(1+ay)} - \frac{1}{y^2}\log(1+ay)\right] \qquad ...(2)$$

But $$\frac{a}{y(1+ay)} = \frac{a}{y}(1+ay)^{-1} = \frac{a}{y}\{1 - ay + a^2y^2 - \ldots\}$$

$$= \frac{a}{y} - a^2 + a^3y - a^4y^2 + \ldots;$$

and $$\frac{1}{y^2}\log(1+ay) = \frac{1}{y^2}\left\{ay - \frac{a^2y^2}{2} + \frac{a^3y^3}{3} - \ldots\right\}$$

$$= \frac{a}{y} - \frac{a^2}{2} + \frac{a^3y}{3} - \ldots$$

On subtraction, $$\frac{a}{y(1+ay)} - \frac{1}{y^2}\log(1+ay)$$

$$= \left(-a^2 + \frac{a^2}{2}\right) + y\left(a^3 - \frac{a^3}{3}\right) + ..$$

$$= -\frac{a^2}{2} + \frac{2a^3}{3}y + \text{ higher powers of } y.$$

$\therefore$ From (2), $$\frac{du}{dy} = (1+ay)^{1/y}\left\{-\frac{a^2}{2} + \frac{2a^3}{3}y + \text{higher powers of } y\right\}$$

Hence from (1) the required limit

$$= \lim_{y\to 0}(1+ay)^{1/y}\left\{-\frac{a^2}{2} + \frac{2}{3}a^3y + \text{higher powers of } y\right\}$$

$$= e^a \times \left(-\frac{a^2}{2}\right) = -\frac{a^2}{2}e^a.$$

EXAMPLE 9. *Evaluate* $\lim_{x\to 0}\dfrac{(1+x)^{1/x} - e}{x}$.

(B.U. 62S, 73A; P.U. 57S, 61A, 64A, 95H; M.U. 71A, 75A, 81A, 87, 90; Kanpur 86)

SOLUTION : When $x = 0$ the given expression $= \dfrac{e-e}{0} = \dfrac{0}{0}$.

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0}\frac{\text{d.c. of } \{(1+x)^{1/x} - e\}}{1};$$

by using L' Hospital's rule

$$= \lim_{x\to 0}\{\text{d.c. of } (1+x)^{1/x}\} \qquad ...(1)$$

Now, to evaluate the d.c. of $(1+x)^{1/x}$, we put

$u = (1+x)^{1/x}$ so that $\log u = \frac{1}{x}\log(1+x)$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{1}{x}\cdot\frac{1}{1+x} - \frac{1}{x^2}\log(1+x)$$

$$= \frac{1}{x}\{1 - x + x^2 - x^3 + x^4 - \ldots\} - \frac{1}{x^2}\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\}$$

$$= -\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots$$

$$\Rightarrow \quad \frac{du}{dx} = (1+x)^{1/x}\left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\} \qquad \ldots(2)$$

Therefore, the required limit, from (1), is

$$= \underset{x\to 0}{Lt}(1+x)^{1/x}\left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\}$$

$$= e \times (-\tfrac{1}{2}) = -\frac{e}{2}.$$

Cor. $\underset{x\to 0}{Lt}\dfrac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \underset{x\to 0}{Lt}\dfrac{\text{d.c. of}\left\{(1+x)^{1/x} - e + \frac{ex}{2}\right\}}{2x}$;

by using L' Hospital's rule

$$= \underset{x\to 0}{Lt}\frac{\text{d.c. of } (1+x)^{1/x} + e/2}{2x}$$

$$= \underset{x\to 0}{Lt}\frac{(1+x)^{1/x}\left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\} + \frac{e}{2}}{2x};$$

from (2)

This is again of the form $\left[\frac{0}{0}\right]$.

Hence again, the limit

$$= \underset{x\to 0}{Lt}\frac{\text{d.c. of}\left[(1+x)^{1/x}\left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\}\right]}{2} \qquad \ldots(3)$$

by using L' Hospitals' rule.

But d.c. of $\left[(1+x)^{1/x}\left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\}\right]$

$$= (1+x)^{1/x}\left\{\frac{2}{3} - \frac{3}{2}x + \ldots\right\} + \left\{-\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \ldots\right\}^2(1+x)^{1/x}$$

Therefore from (3), the required limit

$$= \frac{e.\frac{2}{3} + \frac{1}{4}.e}{2} = \frac{11}{24}e.$$

Thus $\underset{x \to 0}{Lt} \dfrac{(1+x)^{1/x} - e + \dfrac{ex}{2}}{x^2} = \dfrac{11}{24}e.$ **(P.U. 56H; M.U. 69A, 78H, 86, 92H)**

Example 12 (A)

Evaluate : $\left[\textbf{Form } \frac{0}{0}\right]$

1. (i) $\underset{x \to 1}{Lt} \dfrac{\log x}{x-1}$ (ii) $\underset{x \to 1}{Lt} \dfrac{\log x}{x^n - 1}$

2. $\underset{x \to 0}{Lt} \dfrac{(1+x)^n - 1}{x}$ **3.** $\underset{x \to 0}{Lt} \dfrac{1 - \sqrt{1-x^2}}{x^2}$

4. $\underset{x \to \frac{\pi}{4}}{Lt} \dfrac{1 - \tan x}{1 - \sqrt{2}\sin x}$ **5.** $\underset{x \to 0}{Lt} \dfrac{x \log x}{1 - \cos x}$

6. (i) $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x}}{\log(1+x)}$ (ii) $\underset{x \to 0}{Lt} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$

7. $\underset{x \to 0}{Lt} \dfrac{e^x + e^{-x} - 2}{x^2}$ **(B.U. 66; M.U. 74)**

8. (i) $\underset{x \to 0}{Lt} \dfrac{x - \sin x}{x^3}$ **(P.U. 79A; R.U. 84)**

(ii) $\underset{x \to 0}{Lt} \dfrac{x - \tan x}{x^3}$ **(M.U. 67A)**

9. $\underset{x \to 0}{Lt} \dfrac{x - \sin x}{\tan^3 x}$ **10.** $\underset{x \to 0}{Lt} \dfrac{\tan x - x}{x^2 \tan x}$ **(P.U. 45S, 56A)**

11. $\underset{x \to 0}{Lt} \dfrac{\log(1+x^3)}{\sin^3 x}$

12. (i) $\underset{x \to 0}{Lt} \dfrac{\sin bx - \sin x}{x^3}$ **(P.U. 55A)**

(ii) $\underset{x \to 0}{Lt} \dfrac{\sin x + \sin bx - 2x}{x^5}$ **(R.U. 65A)**

13. $\underset{x \to 0}{Lt} \dfrac{\cos bx - \cos x}{x \sin x}$ **(R.U. 64S; P.U. 69A, 72A)**

14. $\underset{x \to 0}{Lt} \dfrac{\tan x - \sin x}{\sin^3 x}$ **15.** $\underset{x \to 0}{Lt} \dfrac{\sin x - x\cos x}{x - \sin x}$

16. $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{x\sin x - \frac{\pi}{2}}{\cos x}$

17. $\underset{x \to 1}{Lt} \dfrac{a^{\log x} - x}{\log x}$

18. $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{a^{\sin x} - a}{\log \sin x}$

19. $\underset{x \to 0}{Lt} \dfrac{x\cos x - \sin x}{x^3}$

20. $\underset{x \to 0}{Lt} \dfrac{x\cos x - \log(1+x)}{x^2}$ **(P.U. 35A, 48A)**

21. (i) $\underset{x \to 0}{Lt} \dfrac{e^x - \log(e + ex)}{x^2}$ (ii) $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$

(P.U. 63A; M.U. 66S; B.U. 61A; Bhag, 63S; R.U. 73A)

(iii) $\underset{x \to 0}{Lt} \left\{\dfrac{1}{x} - \dfrac{1}{x^2}\log(1+x)\right\}$ **(Gorakhpur 81)**

22. $\underset{x \to 0}{Lt} \dfrac{e^x + \log\frac{1-x}{e}}{\tan x - x}$ **(Gorakhpur 82)**

23. $\underset{x \to 0}{Lt} \dfrac{e^x - 2\cos x + e^{-x}}{x\sin x}$

24. $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$ **(P.U. 69S; B.U. 86)**

25. (i) $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x} - 2x}{x^2\sin x}$ (ii) $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ **(B.U. 71A; M.U. 74A)**

26. $\underset{x \to 0}{Lt} \dfrac{x\sin x - \sin^2 x}{x^4}$ **(P.U. 57S)**

27. $\underset{x \to 0}{Lt} \dfrac{x\sin x + \log(1 - x^2)}{x^4}$ **(P.U. 41S)**

28. $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

29. $\underset{x \to 0}{Lt} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$

30. $\underset{x \to 0}{Lt} \dfrac{a^x - 1 - x\log a}{x^2}$

31. $\underset{x \to 0}{Lt} \dfrac{e^x - e^{x\cos x}}{x - \sin x}$

32. $\underset{x \to b}{Lt} \dfrac{x^b - b^x}{x^x - b^b}$

33. $\underset{a \to b}{Lt} \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}$ **(P.U. 59A)**

34. $\underset{x \to 0}{Lt} \dfrac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$ **(P.U. 68A; M.U. 69A)**

35. $\underset{x \to 0}{Lt} \dfrac{\log\sec x - \frac{1}{2}x^2}{x(x - \sin x)}$ **(P.U. 55S)**

36. $\underset{x \to 0}{Lt} \dfrac{\sin x \sin^{-1} x}{x^2}$

37. $\underset{x \to 0}{Lt} \dfrac{\sin x \sin^{-1} x - x^2}{x^6}$

38. $\underset{x \to 0}{Lt} \dfrac{\log(1 - x)}{\tan x}$ **(P.U. 58A)**

39. If $\underset{x \to 0}{Lt} \dfrac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and the limit. **(P.U. 35S, 64A)**

40. Find the values of a and b so that $\underset{x \to 0}{Lt} \dfrac{(1 + a x \sin x) - b\cos x}{x^4}$ may tend to a definite limit and determine it. **(P.U. 65S)**

ANSWERS

1. (i) 1 (ii) $\frac{1}{n}$ **2.** n **3.** $\frac{1}{2}$ **4.** 2 **5.** 2 **6.** (i) 2 (ii) $\frac{2a}{b}$ **7.** 1 **8.** (i) $\frac{1}{6}$ (ii) $-\frac{1}{3}$ **9.** $\frac{1}{6}$ **10.** $\frac{1}{3}$ **11.** 1 **12.** (i) $\frac{1}{3}$ (ii) $\frac{1}{60}$ **13.** 1 **14.** $\frac{1}{2}$ **15.** 2 **16.** -1 **17.** $\log a - 1$ **18.** $a\log a$ **19.** $-\frac{1}{3}$ **20.** $\frac{1}{2}$ **21.** (i) 1 (ii) $\frac{3}{2}$ (iii) $\frac{1}{2}$ **22.** $-\frac{1}{2}$ **23.** 2 **24.** 1 **25.** (i) $\frac{1}{3}$ (ii) 2 **26.** $\frac{1}{6}$ **27.** $-\frac{2}{3}$ **28.** $-\frac{2}{3}$ **29.** 4 **30.** $\frac{1}{2}(\log a)^2$ **31.** 3 **32.** $\dfrac{1 - \log b}{1 + \log b}$ **33.** $b^{x-1}(b\cos bx - \sin bx)\cos^2 bx$ **34.** $\frac{11e}{24}$ **35.** $\frac{1}{2}$ **36.** 1 **37.** $\frac{1}{18}$ **38.** ∞ **39.** $a = -2$, limit $= -1$ **40.** $a = -\frac{1}{2}$, $b = 1$, limit $= \frac{1}{24}$.

II. Form $\frac{\infty}{\infty}$.

Let $\phi(a) = \infty$, $\psi(a) = \infty$, then $\dfrac{\phi(x)}{\psi(x)}$ is of the form $\dfrac{\infty}{\infty}$ when $x \to a$.

In order to bring it to the form $\frac{0}{0}$, we write it as follows.

$$\frac{\phi(x)}{\psi(x)} = \frac{\frac{1}{\psi(x)}}{\frac{1}{\phi(x)}}.$$

When it is reduced to the form $\frac{0}{0}$, then we can find out the limit by the method given in form I. Otherwise, as per situation, we can find out the limit exactly as in form I. The statement of that limit is as follows:

If $\underset{x \to a}{Lt}\ \phi(x) = \infty$ *and* $\underset{x \to a}{Lt}\ \psi(x) = \infty$, *then*

$$\underset{x \to a}{Lt}\ \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}$$

provided that the limit exists, finite or infinite.

We have, $\underset{x \to a}{Lt}\ \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt}\ \frac{1/\psi(x)}{1/\phi(x)}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \underset{x \to a}{Lt}\ \frac{-\frac{\psi'(x)}{[\psi(x)]^2}}{-\frac{\phi'(x)}{[\phi(x)]^2}}$$ [L' Hospital's Rule]

$$= \underset{x \to a}{Lt}\left[\frac{\psi'(x)}{\phi'(x)} \cdot \left(\frac{\phi(x)}{\psi(x)}\right)^2\right]$$

$$= \underset{x \to a}{Lt}\ \frac{\psi'(x)}{\phi'(x)} \cdot \left[\underset{x \to a}{Lt}\ \frac{\phi(x)}{\psi(x)}\right]^2 \qquad \ldots(1)$$

Let $\underset{x \to a}{Lt}\ \frac{\phi(x)}{\psi(x)} = k.$

Now three cases arise :—

Case (i) If $k \neq 0$ and $k \neq \infty$, then from (1)

$$k = \underset{x \to a}{Lt}\ \frac{\psi'(x)}{\phi'(x)} \cdot k^2$$

$$\therefore \quad k = \underset{x \to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}.$$

Case (ii) If $k = 0$, then adding (1) to both sides

$$k + 1 = \underset{x \to a}{Lt}\ \frac{\phi(x) + \psi(x)}{\psi(x)}$$

$$= \underset{x \to a}{Lt}\ \frac{\phi'(x) + \psi'(x)}{\psi'(x)};$$ [case (i)]

$$= \underset{x \to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)} + 1$$

$$\therefore \quad k = \underset{x \to a}{Lt}\ \frac{\phi'(x)}{\psi'(x)}.$$

Case (iii) If $k = \infty$, then

$$\lim_{x \to a} \frac{1}{[\phi(x)/\psi(x)]} = \lim_{x \to a} \frac{\psi(x)}{\phi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \quad \text{[case (ii)]}$$

$$\therefore \quad \lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Hence it is proved in all the three cases that

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

EXAMPLE 1. *Evaluate* $\lim_{x \to \infty} \frac{x^n}{e^x}$. **(P.U. 75A)**

SOLUTION : This is of the form $\frac{\infty}{\infty}$.

$$\therefore \text{ The given limit} = \lim_{x \to \infty} \frac{nx^{n-1}}{e^x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \infty} \frac{n(n-1)x^{n-2}}{e^x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \text{.....}$$

$$= \lim_{x \to \infty} \frac{n(n-1)(n-2) \ldots 3.2.1}{e^x}$$

$$= \lim_{x \to \infty} \frac{n!}{e^x} = \frac{n!}{\infty} = 0.$$

EXAMPLE 2. *Evaluate* $\lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)}$.

(B.U. 59A, 69A, 74A; Bhag. 64A; M.U. 85)

SOLUTION : This is of the form $\frac{\infty}{\infty}$.

By applying L' Hospital rule, we get the given limit

$$= \lim_{x \to a} \frac{\frac{1}{x-a}}{\frac{1}{e^x - e^a} \times e^x}$$

$$= \lim_{x \to a} \frac{1}{x-a} \times \frac{e^x - e^a}{e^x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to a} \frac{e^x}{(x-a)e^x + e^x(1)} = \frac{e^a}{e^a} = 1.$$

EXAMPLE 3. *Evaluate* $\underset{x \to 0}{Lt} \log_{\tan x}(\tan 2x)$. **(P.U. 63A; R.U. 69A, 72A)**

SOLUTION : The given limit $= \underset{x \to 0}{Lt} \dfrac{\log_e \tan 2x}{\log_e \tan x}$;

since by the change of base formula, we have $\log_a m = \dfrac{\log_e m}{\log_e a}$.

This is of the form $\dfrac{\infty}{\infty}$.

Hence by L' Hospital's rule, we have the given limit

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{\tan 2x} \times \sec^2 2x.2}{\frac{1}{\tan x} \times \sec^2 x}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\cos^2 2x} \cdot 2}{\frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}}$$

$$= \underset{x \to 0}{Lt} \frac{2}{\sin 2x \cos 2x} \times \sin x \cos x = \underset{x \to 0}{Lt} \frac{\sin 2x}{\sin 2x \cos 2x}$$

$$= \underset{x \to 0}{Lt} \frac{1}{\cos 2x} = 1.$$

Example 12 (B)

$\left[\text{Form } \dfrac{\infty}{\infty}\right]$

Evaluate:—

1. $\underset{n \to \infty}{Lt} \dfrac{\log n}{n}$ 2. $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{\tan x}{\tan 3x}$ 3. $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{\sec x}{\sec 3x}$

4. $\underset{x \to 0}{Lt} \dfrac{\log x}{\cot x}$ **(M.U. 82H; Kanpur 88)**

5. $\underset{x \to 0}{Lt} \dfrac{\log x^2}{\log \cot^2 x}$ **(P.U. 31S)** 6. $\underset{x \to 0}{Lt} \dfrac{\log \sin 2x}{\log \sin x}$.

7. $\underset{x \to 0}{Lt} \log_x \tan x$ [Use $\log_x m = \log_e m / \log_e x$]

8. $\underset{x \to 0}{Lt} \log_{\sin x}(\sin 2x)$ **(M.U. 73A, 80A)**

9. $\underset{x \to 0}{Lt} \log_{\tan^2 x}(\tan^2 2x)$ **(R.U. 69A, 82; M.U. 72A, 91H)**

ANSWERS

1. 0 **2.** 3 **3.** −3 **4.** 0 **5.** −1 **6.** 1 **7.** 1 **8.** 1 **9.** 1

III. Form $0 \times \infty$

Let $u = \phi(x) \times \psi(x)$

where $\phi(x) \to 0$ and $\psi(x) \to \infty$ when $x \to a$.

Then $\phi(x) \times \psi(x) \to 0 \times \infty$ when $x \to a$ (indeterminate).

In order to bring it to the form $\frac{0}{0}$ we write $\phi(x) \times \psi(x)$ in the following way:

$$u = \frac{\phi(x)}{\frac{1}{\psi(x)}}.$$

When it is reduced to the form $\frac{0}{0}$, $\left(\text{or } \frac{\infty}{\infty}\right)$ then we find out its limit by the method given in form I. (or form II).

EXAMPLE 1. *Evaluate* $\lim_{x \to \infty} 2^x \tan\frac{a}{2^x}$. **(P.U. 52A, 64S)**

SOLUTION : This is of the form $\infty \times 0$.

Let $\frac{1}{2^x} = \theta$, so that when $x \to \infty$, then $\theta \to 0$.

$\therefore$ The given limit $= \lim_{\theta \to 0} \frac{1}{\theta} \tan a\theta$

$$= a \lim_{\theta \to 0} \frac{\tan a\theta}{a\theta} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= a \times 1 = a.$$

EXAMPLE 2. *Evaluate* $\lim_{x \to 0} \sin x \log x$. **(B.U. 56A)**

SOLUTION : This is of the form $0 \times \infty$ when $x \to 0$.

$\therefore$ The given limit $= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x \to 0} \frac{-\sin x.\tan x}{x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-(\sin x \sec^2 x + \tan x \cos x)}{1}$$

$$= 0.$$

Example 12 (C)

[Form 0 × ∞]

Evaluate:—

1. $\lim_{x \to 0} x \log x$ **(P.U. 61S; B.U. 63A; Kanpur 85)**

2. $\lim_{x \to 0} x \log \tan x$

3. $\lim_{x \to 1} (1 - x)\tan\frac{\pi x}{2}$

4. $\lim_{x \to 1} \sec\frac{\pi}{2x}\log x$

5. $\lim_{x \to 0} x^m(\log x)^n$ where m and n are positive integers. **(B.U. 69S)**

6. $\lim_{x \to 0} \sqrt{a^2 - x^2} \cot\left[\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right]$ **(B.U. 53S)**

Answers

1. 0 2. 0 3. $\frac{2}{\pi}$ 4. $\frac{2}{\pi}$ 5. 0 6. $\frac{4a}{\pi}$.

IV. Form ∞—∞

Let $u = \phi(x) - \psi(x)$

where $\phi(x) \to \infty$ and $\psi(x) \to \infty$ when $x \to a$.

Then $\phi(x) - \psi(x) \to \infty - \infty$ (indeterminate) when $x \to a$.

To reduce this to the form $\frac{0}{0}$ we write $\phi(x) - \psi(x)$ as follows:

$$\phi(x) - \psi(x) = \frac{\frac{1}{\psi(x)} - \frac{1}{\phi(x)}}{\frac{1}{\phi(x).\psi(x)}}$$

Now its limit can be determined according to form I.

Worked out examples

EXAMPLE 1. *Find* $\lim_{x \to 0} \frac{1}{x} - \cot x$.

(B.U. 64A; P.U. 51A; Bhag.63A; Mithila 80A)

SOLUTION : Putting $x = 0$ in the given expression, it reduces to the form $\infty - \infty$ which is indeterminate.

Let $y = \frac{1}{x} - \cot x$.

Then $y = \frac{1}{x} - \frac{1}{\tan x}$

$$= \frac{\tan x - x}{x \tan x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \frac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right) - x}{x\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right)}$$

$$= \frac{\frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}{x^2 + \frac{x^4}{3} + \ldots} = \frac{\frac{x}{3} + \frac{2}{15}x^3 + \ldots}{1 + \frac{x^2}{3} + \ldots}$$

$$\therefore \lim_{x \to 0} y = 0.$$

Cor. $$\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right) = \lim_{x \to 0} \frac{1}{x} \cdot \frac{\frac{x}{3} + \frac{2}{15}x^3 + \ldots}{1 + \frac{x^2}{3} + \ldots}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} + \frac{2}{15}x^2 + \ldots}{1 + \frac{x^2}{3} + \ldots}$$

$$= \frac{1}{3}.$$

i.e. $$\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{x \tan x}\right) = \frac{1}{3}.$$ **(B.U. 54A; Bhag. 68A; Kanpur 86)**

EXAMPLE 2. *Evaluate* $\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

(R.U. 66A, 68S, 76A, 81; Bhag. 63S, 69A; B.U. 56S; P.U. 44A, 49A, 54A, 77A)

SOLUTION : This is of the form $\infty - \infty$.

$\therefore$ The given limit $= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}$ $\qquad \left[\text{Form } \frac{0}{0}\right]$

$$= \underset{x \to 0}{Lt} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2 - x^2}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2}$$

$$= \underset{x \to 0}{Lt} \frac{x^2 - \frac{2x^4}{6} + x^6\left(\frac{1}{36} + \frac{2}{120}\right) + \ldots - x^2}{x^2\left\{x^2 - \frac{2x^4}{6} + x^6\left(\frac{1}{36} + \frac{2}{120}\right) + \ldots.\right\}}$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{x^4}{3} + x^6\left(\frac{1}{36} + \frac{1}{60}\right) + \ldots}{x^4 - \frac{x^6}{3} + \ldots}$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{1}{3} + x^2\left(\frac{1}{36} + \frac{1}{60}\right) + \ldots}{1 - \frac{x^2}{3} + \ldots} = -\frac{1}{3}.$$

Note : Since the problem is of the form $\frac{0}{0}$, therefore we can work it out by using the formula

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi(x)}.$$

But it will be very lengthy.

Example 12 (D)

[Form $\infty - \infty$]

Evaluate:—

1. $\underset{x \to \frac{\pi}{2}}{Lt} (\sec x - \tan x)$.

2. $\underset{x \to \frac{\pi}{2}}{Lt} \left(x \tan x - \frac{\pi}{2} \sec x\right)$. **(P.U. 37S, 46S, 93H; Bhag. 62S, 70A; B.U. 70A)**

3. $\underset{x \to 0}{Lt} \left\{\frac{1}{2x^2} - \frac{1}{2x \tan x}\right\}$. **(B.U. 54A; P.U. 39A; Bhag. 68A)**

[Hints : The given limit $= \underset{x \to 0}{Lt} \left\{\frac{1}{2x^2} - \frac{\cos x}{2x \sin x}\right\}$

$= \underset{x \to 0}{Lt} \left\{\frac{\sin x - x\cos x}{2x^2 \sin x}\right\}$ etc.]

4. $\underset{x\to 0}{Lt}\ \dfrac{\cot x-\dfrac{1}{x}}{x}$

5. $\underset{x\to 0}{Lt}\left(\dfrac{1}{x^2}-\cot^2 x\right)$

6. $\underset{x\to 0}{Lt}\left(\dfrac{1}{e^x-1}-\dfrac{1}{x}\right)$

7. $\underset{x\to 0}{Lt}\left\{\dfrac{1}{x}-\dfrac{1}{x^2}\log(1+x)\right\}$

8. $\underset{x\to 1}{Lt}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$ **(B.U. 60S)**

ANSWERS

1. 0 **2.** -1 **3.** $\frac{1}{6}$ **4.** $-\frac{1}{3}$ **5.** $\frac{2}{3}$ **6.** $-\frac{1}{2}$ **7.** $\frac{1}{2}$ **8.** $\frac{1}{2}$.

V. Form 0°, ∞° and 1^∞.

Let $y = u^v$ ...(1)

where u and v are functions of x.

Now taking the logarithm of (1), we get

$$\log y = v \log u \qquad ...(2)$$

case (i) If $u = 0$, $v = 0$ when $x = a$, then $y = 0^\circ$

case (ii) If $u = \infty$, $v = 0$ " " " $y = \infty^\circ$

case (iii) If $u = 1$, $v = \infty$ " " " $y = 1^\infty$.

Then from (2),

according to case (i) if $u = 0$, $v = 0$, then $\log y = \infty \times 0$

" " (ii) if $u = \infty$, $v = 0$, then $\log y = 0 \times \infty$

" " (iii) if $u = 1$, $v = \infty$, then $\log y = \infty \times 0$.

That is, in all the three situations, $\log y$ is reduced to the indeterminate form $0 \times \infty$.

Hence in these cases, we shall first of all take the logarithm of the given function and then proceed according to Form III.

EXAMPLE *Limit of* $\left(1+\dfrac{a}{x}\right)^x$ *as* $x \to \infty$. **(P.U. 76A, 90H; M.U. 89H; Kanpur 88)**

SOLUTION : Let $y = \underset{x\to\infty}{Lt}\left(1+\dfrac{a}{x}\right)^x$.

Then $\log y = \underset{x\to\infty}{Lt}\ x\log\left(1+\dfrac{a}{x}\right)$.

$$= \underset{x\to\infty}{Lt}\ \frac{\log\left(1+\dfrac{a}{x}\right)}{1/x} \qquad \left[\text{Form}\ \frac{0}{0}\right]$$

$$= \underset{x \to \infty}{Lt} \frac{\frac{1}{1 + a/x} \cdot (-a/x^2)}{-1/x^2} = \underset{x \to \infty}{Lt} \frac{a}{1 + a/x} = a.$$

$\therefore \quad y = e^a.$

$\therefore$ The required limit $= e^a$.

Second Method :

$$\log y = \underset{x \to \infty}{Lt} x \log\left(1 + \frac{a}{x}\right)$$

$$= \underset{x \to \infty}{Lt} x\left[\frac{a}{x} - \frac{1}{2} \cdot \frac{a^2}{x^2} + \frac{1}{3} \cdot \frac{a^3}{x^3} - \ldots\right]$$

$$= \underset{x \to \infty}{Lt}\left[a - \frac{1}{2} \cdot \frac{a^2}{x} + \frac{1}{3} \cdot \frac{a^3}{x^2} + \ldots\right] = a.$$

$\therefore \quad y = e^a.$

Cor. Putting $a = 1$, we have $\underset{x \to \infty}{Lt}\left(1 + \frac{1}{x}\right)^x = e.$

Now replacing x by $\frac{1}{y}$ so that when $x \to \infty$, $y \to 0$, we get

$$\underset{y \to 0}{Lt} (1 + y)^{1/y} = e.$$

Hence $\underset{x \to 0}{Lt} (1 + x)^{1/x} = e.$ **(B.U. 91H; Gorakhpur 87)**

Worked out Examples

EXAMPLE 1. *Evaluate* $\underset{x \to 1}{Lt} x^{\frac{1}{1-x}}$.

SOLUTION : This is of the form 1^∞.

Let $\quad y = \underset{x \to 1}{Lt} x^{\frac{1}{1-x}}.$

$\therefore \quad \log y = \underset{x \to 1}{Lt} \frac{1}{1-x} \cdot \log x$

$\therefore \quad \log y = \underset{x \to 1}{Lt} \frac{\log x}{1-x}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \underset{x \to 1}{Lt} \frac{\frac{1}{x}}{-1} = -1.$$

$\therefore \quad y = e^{-1}.$

$\therefore$ The reqd. limit $= e^{-1}$.

EXAMPLE 2. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

(P.U. 31S, 47A, 92H; M.U. 64A; R.U. 69S; B.U. 72A, 87)

SOLUTION : Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

$\therefore \quad \log y = \frac{1}{x^2} \log \frac{\tan x}{x}$. ...(1)

But $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

$\therefore \quad \frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots$

$\therefore \quad \log \frac{\tan x}{x} = \log \left\{1 + \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)\right\}$

$$= \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \frac{1}{3}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^3 + \ldots$$

$$= \frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{2} \cdot \frac{1}{9}\right) + \ldots$$

$$= \frac{x^2}{3} + \frac{7}{90}x^4 + \text{higher powers of } x^2.$$

Therefore from (1),

$$\log y = \frac{1}{x^2}\left(\frac{x^2}{3} + \frac{7}{90}x^4 + \ldots\right)$$

$$= \frac{1}{3} + \frac{7}{90}x^2 + \text{higher powers of } x^2$$

$\therefore \quad \underset{x \to 0}{Lt} \log y = \frac{1}{3} \quad \therefore \quad \underset{x \to 0}{Lt}\, y = e^{\frac{1}{3}}$.

EXAMPLE 3. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$.

(R.U. 67A; Bhag. 66A, 69S; B.U. 58S, 65A, 67S; M.U. 63A, 72S, 76; P.U. 50A, 59S, 60S, 80A; Kanpur 85)

SOLUTION : Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$. $\quad \therefore \quad \log y = \frac{1}{x} \log \frac{\tan x}{x}$.

But $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

$\therefore \quad \frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots$

Then like the previous exercise,

$$\log y = \frac{1}{x}\left(\frac{x^2}{3} + \frac{7}{90}x^4 + \ldots.\right)$$

$$= \frac{x}{3} + \frac{7}{90}x^3 + \ldots$$

$\therefore \quad \underset{x \to 0}{Lt} \log y = 0 \quad \therefore \quad Lt\, y = e^0 = 1.$

EXAMPLE 4. *Evaluate (i)* $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

(Bhag. 62A; B.U. 53A, 61S, 80A, 85; M.U. 66A; R.U. 75A, 85; P.U. 86)

(ii) $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$. **(R.U. 63A; B.U. 66A, 90H; Gorakhpur 84)**

SOLUTION : (i) Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

$$\therefore \quad \log y = \frac{1}{x^2} \log \frac{\sin x}{x} \qquad \ldots(1)$$

But $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$$\therefore \quad \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

$$\therefore \log \frac{\sin x}{x} = \log\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)$$

$$= \log\left\{1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)\right\}$$

$$= -\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)^2$$

$$+ \frac{1}{3}\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)^3 + \ldots$$

$$= -\frac{x^2}{6} + x^4\left(\frac{1}{120} - \frac{1}{2} \cdot \frac{1}{36}\right) + \ldots$$

$$= -\frac{x^2}{6} - \frac{x^4}{180} + \ldots$$

$\therefore$ From (1),

$$\log y = \frac{1}{x^2}\left[-\frac{x^2}{6} - \frac{x^4}{180} + \ldots\right]$$

$$= -\frac{1}{6} - \frac{x^2}{180} + \text{higher powers of } x^2$$

$$\therefore \lim_{x \to 0} \log y = -\frac{1}{6}, \quad \therefore \lim_{x \to 0} y = e^{-\frac{1}{6}}.$$

(ii) Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

$$\therefore \quad \log y = \frac{1}{x} \log \frac{\sin x}{x}.$$

Then like (1),

$$\log y = \frac{1}{x}\left[-\frac{x^2}{6} - \frac{x^4}{180} + \ldots\right]$$

$$= -\frac{x}{6} - \frac{x^3}{180} + \ldots$$

$$\therefore \lim_{x \to 0} \log y = 0, \quad \therefore \lim_{x \to 0} y = e^0 = 1.$$

EXAMPLE 5. *Evaluate* $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

(R.U. 66A, 68A, 72A; Bhag. 65A; B.U. 55A, 57A, 62A, 68A; M.U. 64S, 90H; P.U. 38S, 49S, 50S, 51A, 62S, 71A)

SOLUTION : Putting $x = \frac{\pi}{2}$ in the given expression, it reduces to the form 1^{∞} which is indeterminate.

Let $y = (\sin x)^{\tan x}$.

$$\therefore \quad \log y = \tan x \log \sin x$$

$$= \frac{\log \sin x}{\cot x} \left(\text{when } x = \frac{\pi}{2} \text{ it is } = \frac{0}{0}\right)$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \log y = \lim_{x \to \frac{\pi}{2}} \frac{\text{d.c of } \log \sin x}{\text{d.c. of } \cot x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{-\text{cosec}^2 x} = \frac{0}{-1} = 0.$$

$$\therefore \quad Lt\, y = e^0 = 1.$$

EXAMPLE 6. *Evaluate* $\lim_{x \to 0} (\cos x)^{\cot^2 x}$.

(B.U. 54S, 57S, 58A, 63S; Bhag. 65S, 66S; P.U. 33A, 45A, 53A, 54S, 66A, 68S, 87A, 94H; Gorakhpur 86)

SOLUTION : When $x = 0$, the given expression $= 1^{\infty}$.

Let $y = (\cos x)^{\cot^2 x}$.

$\therefore \quad \log y = \cot^2 x \log \cos x$

$$= \frac{\log \cos x}{\tan^2 x} \quad \left(\text{which is } = \frac{0}{0} \text{ when } x = 0\right)$$

$$\therefore \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\text{d.c. of } \log \cos x}{\text{d.c. of } \tan^2 x} = \lim_{x \to 0} \frac{\frac{1}{\cos x} \times (-\sin x)}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \sec^2 x} = \lim_{x \to 0} -\frac{1}{2 \sec^2 x} = -\frac{1}{2}$$

$$\therefore \quad \lim_{x \to 0} y = e^{-1/2}.$$

EXAMPLE 7. *Evaluate* $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$

(Bhag. 63A; P.U. 46A, 64A; R.U. 71A; M.U. 80H)

SOLUTION : Putting $x = a$ in the given expression it reduces to the form 1^{∞} which is indeterminate.

Let $y = \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$

$$\therefore \quad \log y = \tan \frac{\pi x}{2a} \log \left(2 - \frac{x}{a}\right)$$

$$= \frac{\log \left(2 - \frac{x}{a}\right)}{\cot \frac{\pi x}{2a}}; \text{ which is } = \frac{0}{0} \text{ when } x = a.$$

$$\therefore \quad \lim_{x \to a} \log y = \lim_{x \to a} \frac{\text{d.c. of } \log \left(2 - \frac{x}{a}\right)}{\text{d.c. of } \cot \frac{\pi x}{2a}}$$

$$= \lim_{x \to a} \frac{\frac{1}{2 - \frac{x}{a}} \times \left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2 \frac{\pi x}{2a} \times \frac{\pi}{2a}} = \frac{-\frac{1}{a}}{-\operatorname{cosec}^2 \frac{\pi}{2} \times \frac{\pi}{2a}};$$

putting $x = a$

$$= \frac{1}{a} \times \frac{2a}{\pi} = \frac{2}{\pi}. \qquad \therefore \quad \lim_{x \to a} y = e^{2/\pi}.$$

EXAMPLE 8. *Evaluate* $\lim_{x \to 0} (\cot x)^{\frac{1}{\log x}}$ **(P.U. 58S, 61A, 88)**

SOLUTION : This is of the form ∞^0.

Let $y = \lim_{x \to 0} (\cot x)^{\frac{1}{\log x}}$

$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{\log x} \cdot \log \cot x$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{\frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x)}{\frac{1}{x}} = \lim_{x \to 0} \frac{-x}{\sin x \cos x}$$

$$= \lim_{x \to 0} -\frac{2x}{\sin 2x} = -1.$$

$\therefore \quad y = e^{-1}$

Example 12 (E)

[Form 0^0, 1^∞, ∞^0]

Evaluate:—

1. $\lim_{x \to 0} x^x$ **(P.U. 53)**
2. $\lim_{x \to p} (x-p)^{x-p}$ **(B.U. 59S)**
3. $\lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$ **(B.U. 69S)**
4. $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$
5. $\lim_{x \to 0} (\sin x)^{\tan x}$ **(P.U. 40A, 44S, 52S, 71A; Bhag. 65A; M.U. 64S; R.U. 66A, 72A; B.U. 67A, 75A, 78A; Gorakhpur 85)**
6. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{2\tan x}$ **(P.U. 47S)**
7. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$
8. $\lim_{x \to 0} (\cos x)^{\cot x}$ **(Bhag. 71A; Kanpur 83)**
9. $\lim_{x \to 0} (\cos mx)^{\operatorname{cosec}^2 nx}$ **(B.U. 55S)**
10. $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x}$ **(B.U. 60A, 64S)**
11. $\lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x}$ **(M.U. 65A; P.U. 60A)**

12. (i) $\lim_{x \to 0} (1 + \sin x)^{\cot x}$ (ii) $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$ **(Kanpur 84)**

13. $\lim_{x \to 0} \left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$ **14.** $\lim_{x \to 0} (\cos x)^{1/x}$

15. $\lim_{x \to 0} (\cos x)^{1/x^2}$ **(B.U. 62S; P.U. 58S, 70A; M.U. 70A)**

16. $\lim_{x \to 1} x^{\frac{1}{x-1}}$ **17.** $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$

18. $\lim_{x \to 0} (\text{cosec } x)^{\frac{1}{\log x}}$ **19.** $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$

20. Show that $\lim_{x \to \infty} a^x \sin\frac{b}{a^x} = 0$ when $0 < a < 1$

and $\lim_{x \to \infty} a^x \sin\frac{b}{a^x} = b$ when $a > 1$. **(R.U. 70A; Gorakhpur 88)**

ANSWER

1. 1 **2.** 1 **3.** 1 **4.** $\frac{1}{e}$ **5.** 1 **6.** 1 **7.** 1 **8.** 1 **9.** $e^{\frac{m^2}{2n^2}}$ **10.** 1

11. 1 **12.** (i) e (ii) $e^{\,1}$ **13.** $e^{1/6}$ **14.** 1 **15.** $e^{-1/2}$ **16.** e **17.** e **18.** $\frac{1}{e}$

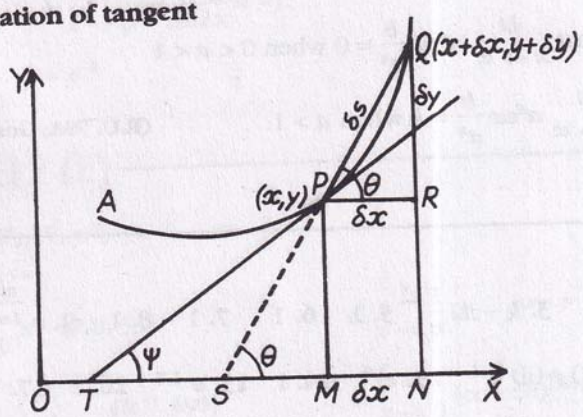
19. $\sqrt{(ab)}$.

5

TANGENTS AND NORMALS

5.1. Tangent : *Definition*—Let P and Q be any two points on a given curve. When the point Q, while moving along the curve, coincides with the point P, then the limiting position of the secant PQ is called the tangent at the point P. That is, a tangent is a straight lime passing through two coincident points.

5.2. Equation of tangent



Let the equation of the curve be $y = f(x)$.

Let P be any point on the curve whose co-ordinates are (x, y). Again, let Q be any other point on the curve which is very very near to P. Therefore, the co-ordinates of Q will be $(x + \delta x, y + \delta y)$.

Let (X, Y) denote the current co-ordinates.

Now the equation of the line PQ is

$$Y - y = \frac{y + \delta y - y}{x + \delta x - x}(X - x)$$

i.e. $$Y - y = \frac{\delta y}{\delta x}(X - x) \qquad ...(1)$$

From P draw PM and from Q draw QN perpendiculars on the x-axis. Then

$$OM = x, \quad ON = x + \delta x$$

$$\therefore \quad MN = \delta x.$$

Now we move the point Q along the curve to meet with the point P. We see that when $Q \to P$, then $\delta x \to 0$ and also chord PQ becomes the tangent at the point P. That is, in order to find the equation of the tangent, we should send $\delta x \to 0$ in the equation (1). That is, the limiting position of the equation (1) when $\delta x \to 0$ gives the equation of the tangent at the point P.

Now taking the limit of (1) as $\delta x \to 0$, we get

$$Y - y = \frac{dy}{dx}(X - x)$$

and this is the equation of the tangent at the point (x, y).

If we take (x, y) as current co-ordinates, then the equation of the tangent at the point (x_1, y_1) will be

$$y - y_1 = \left(\frac{dy}{dx}\right)_{\substack{x_1 \\ y_1}} (x - x_1).$$

Here $\left(\frac{dy}{dx}\right)_{\substack{x_1 \\ y_1}}$ means that first of all we shall evaluate $\frac{dy}{dx}$ from the given function and then we shall put $x = x_1$ and $y = y_1$ in the diff. coeff.

Note : The slope of the tangent is given by $\frac{dy}{dx}$ $(= \tan\psi)$. Thus if the tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$.

Second Form : Let the equation of the curve be given by $f(x, y) = 0$.

We know from the chapter on partial differentiation that

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Therefore putting the value of $\frac{dy}{dx}$ in the equation

$$Y - y = \frac{dy}{dx}(X - x)$$

we get

$$Y - y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial f}}(X - x)$$

or $$(Y-y)\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x}(X-x)$$

$$\therefore \quad (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} = 0.$$

This is the second form of the equation of the tangent.

5.3. Equation of the normal

Definition—The normal at any pont (x, y) is that st. line which is perpendicular to the tangent at (x, y).

Thus to find out the equation of the normal at the point (x, y), we shall have to find out the equation of that straight line which passes through (x, y) and is perpendicular to the tangent at (x, y). We, therefore assume that the equation of the normal is

$$Y - y = m(X - x). \qquad ...(1)$$

We know from the last article that the equation of the tangent at the point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(2)$$

Now since (1) and (2) are perpendicular to each other, therefore

$$m \times \left(\frac{dy}{dx}\right) = -1, \quad \text{i.e.} \quad m = -\frac{1}{\frac{dy}{dx}}.$$

$\therefore$ Putting $m = -\dfrac{1}{\frac{dy}{dx}}$ in (1), we get

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

$$\Rightarrow \quad \frac{dy}{dx}(Y - y) = -(X - x)$$

$$\Rightarrow \quad (X - x) + (Y - y)\frac{dy}{dx} = 0.$$

This is the equation of the normal at the point (x, y).

Second form : As before, since

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}},$$

$\therefore$ Putting the value of $\frac{dy}{dx}$ in the equation of the normal $(X - x) + (Y - y)\frac{dy}{dx} = 0$, we get

$$(X - x) - (Y - y)\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = 0.$$

$$\Rightarrow \quad X - x = (Y - y)\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\therefore \quad \frac{X - x}{\frac{\partial f}{\partial x}} = \frac{Y - y}{\frac{\partial f}{\partial y}}.$$

This is the second form of the equation of the normal.

EXAMPLE *Find the equation of the tangent and normal at the point* (α, β) *of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \textbf{(R.U.: 70A)}$$

SOLUTION : Here $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$\therefore$ The equation of the tangent at the point (x, y) is

$$Y - y = -\frac{b^2}{a^2} \cdot \frac{x}{y}(X - x)$$

$$\Rightarrow \quad b^2x(X - x) + a^2y(Y - y) = 0$$

$$\Rightarrow \quad b^2Xx - b^2x^2 + a^2Yy - a^2y^2 = 0.$$

Hence the equation of the tangent at the point (α, β) will be of the form (by putting $x = \alpha$, $y = \beta$ and $X = x$, $Y = y$)

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1.$$

Again, the equation of the normal at the point (x, y) is

$$(X - x) + (Y - y)\left(-\frac{b^2}{a^2}\frac{x}{y}\right) = 0$$

$$\Rightarrow \qquad (X - x)a^2y = b^2x(Y - y).$$

That is, the equation of the normal at the point (α, β) is

$$(x - \alpha)a^2\beta = b^2\alpha(y - \beta) \Rightarrow \frac{a^2(x - \alpha)}{\alpha} = \frac{b^2(y - \beta)}{\beta}.$$

Worked out Examples

EXAMPLE 1. *Find the equation of the tangent at (a, b) to the curve*

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$$

(P.U. 45, 66S, B.U. 61A; R.U. 64, 68S, 69S, 84; M.U. 75A)

SOLUTION : First we shall find $\frac{dy}{dx}$ at the point (a, b) of the given curve.

Given that

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$$

$\therefore$ Differentiating we get

$$n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b}\frac{dy}{dx} = 0.$$

$$\therefore \qquad \left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b}\frac{dy}{dx} = -\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a}$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{x^{n-1}}{a^n} \times \frac{b^n}{y^{n-1}}$$

$\therefore$ at the point (a, b), $\quad \frac{dy}{dx} = -\frac{a^{n-1}}{a^n} \times \frac{b^n}{b^{n-1}} = -\frac{b}{a}.$

$\therefore$ The reqd. equation of the tangent at the point (a, b) is

$$y - b = \left(\frac{dy}{dx}\right)_{\substack{x = a \\ y = b}} (x - a)$$

$$\Rightarrow \qquad y - b = -\frac{b}{a}(x - a)$$

$$\Rightarrow \qquad ay - ab = -bx + ab$$

$$\Rightarrow \qquad bx + ay = 2ab.$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 2; \text{ dividing by } ab;$$

Note : Since the equation of the tangent does not depend on n, therefore the equation of the tangent at the point (a, b) of the given curve, for different values of n, will be always,

$$\frac{x}{a} + \frac{y}{b} = 2 \text{ whether } n = 2, \frac{3}{2} \text{ .....etc.}$$

For example, for $n = 2$, the equation of the tangent at the point (a, b) on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ will be $\frac{x}{a} + \frac{y}{b} = 2$.

Similarly, putting $n = 3/2$, the equation of the tangent at the point (a, b) on the curve $\left(\frac{x}{a}\right)^{3/2} + \left(\frac{y}{b}\right)^{3/2}$ will be $\frac{x}{a} + \frac{y}{b} = 2$ and so on.

EXAMPLE 2. *Find the equation of the tangent to the curve* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ *at the point* θ.

SOLUTION : Let $P(x = a\cos^3\theta, y = a\sin^3\theta)$ be any point on the given curve.

Then $\frac{dy}{d\theta} = a.3\sin^2\theta\cos\theta$, $\frac{dx}{d\theta} = a.3\cos^2\theta(-\sin\theta)$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta.$$

Then the equation of the tangent at P is

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\Rightarrow \quad (y - a\sin^3\theta)\cos\theta = -\sin\theta(x - a\cos^3\theta)$$

$$\Rightarrow \quad y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$$

$$\Rightarrow \quad x\sin\theta + y\cos\theta = a\sin^3\theta\cos\theta + a\sin\theta\cos^3\theta$$

$$= a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= a\sin\theta\cos\theta.$$

EXAMPLE 3. *Write down the equation of the tangent to the curve* $y(x^2 + a^2) = ax^2$ *at the point where* $y = \frac{a}{4}$. **(P.U. 52A; R.U. 65A)**

SOLUTION : Putting $y = \frac{a}{4}$ in the given equation,

$$\frac{a}{4}(x^2 + a^2) = ax^2 \Rightarrow x^2 + a^2 = 4x^2$$

$$\Rightarrow \quad 3x^2 = a^2 \Rightarrow \sqrt{3}x = a \quad \therefore \quad x = \frac{a}{\sqrt{3}}.$$

Thus we have to find the equation of the tangent at the point $\left(\frac{a}{\sqrt{3}}, \frac{a}{4}\right)$.

Given the equation of the curve $y(x^2 + a^2) = ax^2$.

Therefore differentiating

$$y(2x) + (x^2 + a^2)\frac{dy}{dx} = a.2x$$

$$\Rightarrow \quad (x^2 + a^2)\frac{dy}{dx} = 2ax - 2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x(a - y)}{x^2 + a^2}.$$

$$\therefore \text{ at the point } \left(\frac{a}{\sqrt{3}}, \frac{a}{4}\right), \quad \frac{dy}{dx} = \frac{2 \times \frac{a}{\sqrt{3}}\left(a - \frac{a}{4}\right)}{\frac{a^2}{3} + a^2}$$

$$= \frac{2a}{\sqrt{3}} \times \frac{3a}{4} \times \frac{3}{4a^2} = \frac{3\sqrt{3}}{8}.$$

$\therefore$ The reqd. equation of the tangent at the point $\left(\frac{a}{\sqrt{3}}, \frac{a}{4}\right)$ is

$$y - \frac{a}{4} = \frac{3\sqrt{3}}{8}\left(x - \frac{a}{\sqrt{3}}\right) \Rightarrow 8y - 2a = 3\sqrt{3}\left(x - \frac{a}{\sqrt{3}}\right)$$

$$\Rightarrow \quad 8y - 2a = 3\sqrt{3}x - 3a \quad \therefore \quad 3\sqrt{3}x - 8y = a.$$

EXAMPLE 4. *Prove that* $\frac{x}{a} + \frac{y}{b} = 1$ *touches the curve* $y = be^{-x/a}$ *at the point where the curve crosses the axis of y.*

(P.U. 33A, 50A; B.U. 67S; M.U. 85, 89H)

SOLUTION : The x-co-ordinate of the point where the curve cuts the y-axis is 0.

In order to find the y-co-ordinate of that point, putting $x = 0$ in the given equation $y = be^{-x/a}$, we have $y = be^0 = b$.

Thus we have to find the equation of the tangent at the point $(0, b)$.

Given the equation of the curve $y = be^{-x/a}$.

Differentiating this, we get

$$\frac{dy}{dx} = be^{-x/a}\left(-\frac{1}{a}\right) = -\frac{b}{a}e^{-x/a}.$$

$\therefore$ at the point $(0, b)$, $\frac{dy}{dx} = -\frac{b}{a} \cdot e^0 = -\frac{b}{a}$.

Hence the required equation of the tangent at the point $(0, b)$ is

$$y - b = -\frac{b}{a}(x - 0)$$

$\Rightarrow \quad ay - ab = -bx \quad \Rightarrow \quad bx + ay = ab$

$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1.$

Hence the result.

EXAMPLE 5. *Show that all the points on the curve* $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ *at which the tangent is parallel to the axis of* x *lie on a certain parabola.* **(P.U. 31S, 35A)**

SOLUTION : If the tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$.

Differentiating the given equation $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a\left[1 + a\cos\frac{x}{a}\cdot\frac{1}{a}\right] = 4a\left[1 + \cos\frac{x}{a}\right].$$

Since $\frac{dy}{dx} = 0$, therefore

$$1 + \cos\frac{x}{a} = 0 \Rightarrow \cos\frac{x}{a} = -1$$

$\Rightarrow \quad \frac{x}{a} = \pi \quad \therefore \quad x = a\pi.$

Now putting $x = a\pi$ in the given equation

$$y^2 = 4a\left(x + a\sin\frac{x}{a}\right), \text{ we get}$$

$$y^2 = 4a(x + a\sin\pi) \Rightarrow y^2 = 4ax. \bullet$$

Hence the result.

EXAMPLE 6. *Tangents are drawn from the origin to the curve* $y = \sin x$. *Prove that their points of contact lie on* $x^2y^2 = x^2 - y^2$. **(P.U. 56S, 69S)**

SOLUTION : Let $P(\alpha, \beta)$ be any point on the curve $y = \sin x$. Then the equation of the tangent at (α, β) will be

$$y - \beta = \left(\frac{dy}{dx}\right)_{\substack{x=\alpha \\ y=\beta}}(x - \alpha) \qquad \ldots(1)$$

Now, differentiating $y = \sin x$,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}.$$

$\therefore$ At the point (α, β), $\frac{dy}{dx} = \sqrt{1 - \beta^2}$.

Hence from (1), the equation of the tangent is

$$y - \beta = \sqrt{1 - \beta^2}(x - \alpha).$$

Since this line passes through the origin (0, 0), therefore

$$0 - \beta = \sqrt{1 - \beta^2}\,(0 - \alpha)$$

$$\Rightarrow \quad \beta = \alpha\sqrt{1 - \beta^2} \quad \Rightarrow \quad \beta^2 = \alpha^2(1 - \beta^2)$$

$$\Rightarrow \quad \beta^2 = \alpha^2 - \alpha^2\beta^2 \quad \Rightarrow \quad \alpha^2\beta^2 = \alpha^2 - \beta^2.$$

$\therefore$ The locus of (α, β), by putting $\alpha = x$ and $\beta = y$, becomes

$$x^2y^2 = x^2 - y^2.$$

EXAMPLE 7. *If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle θ with the x-axis, show that its equation is $y\cos\theta - x\sin\theta = a\cos 2\theta$.*
(P.U. 60S, 87, 92H; R.U. 71H, 72H; Bhag. 90H; B.U. 91H, 94H)

SOLUTION : From the formula, the equation of the normal is

$$\frac{X - x}{\frac{\partial f}{\partial x}} = \frac{Y - y}{\frac{\partial f}{\partial y}}. \qquad \ldots(1)$$

Let $f(x, y) \equiv x^{2/3} + y^{2/3} = a^{2/3}$.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{2}{3}x^{\frac{2}{3} - 1} = \frac{2}{3x^{1/3}},$$

$$\frac{\partial f}{\partial y} = \frac{2}{3}y^{\frac{2}{3} - 1} = \frac{2}{3y^{1/3}}.$$

$\therefore$ from (1), the equation of the normal becomes

$$\frac{X - x}{\frac{2}{3}/x^{1/3}} = \frac{Y - y}{\frac{2}{3}/y^{1/3}}$$

$$\Rightarrow \quad Y - y = \frac{x^{1/3}}{y^{1/3}}(X - x) \qquad \ldots(2)$$

which we have put in the form $y = mx + c$.

Since the line (2), makes an angle θ with the x-axis, therefore

$$\tan\theta = \frac{x^{1/3}}{y^{1/3}}.$$

$$\Rightarrow \quad \frac{x^{1/3}}{y^{1/3}} = \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \quad \frac{x^{1/3}}{\sin\theta} = \frac{y^{1/3}}{\cos\theta}$$

$$\Rightarrow \quad \frac{x^{1/3}}{\sin\theta} = \frac{y^{1/3}}{\cos\theta} = \frac{\sqrt{x^{2/3} + y^{2/3}}}{\sqrt{\sin^2\theta + \cos^2\theta}} = a.$$

$$\therefore \quad x^{1/3} = a^{1/3}\sin\theta, \quad y^{1/3} = a^{1/3}\cos\theta$$

i.e. $x = a\sin^3\theta, \quad y = a\cos^3\theta.$

Therefore from the equation (2),

$$Y - a\cos^3\theta = \frac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}(X - a\sin^3\theta)$$

$$\Rightarrow \quad Y\cos\theta - a\cos^4\theta = X\sin\theta - a\sin^4\theta$$

or in terms of x, y,

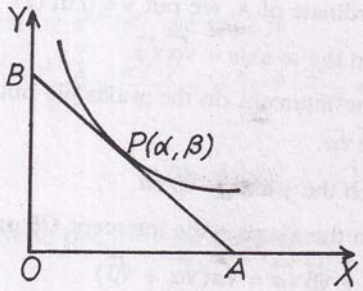
$$y\cos\theta - x\sin\theta = a\cos^4\theta - a\sin^4\theta = a(\cos^4\theta - \sin^4\theta)$$
$$= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$= a\cos 2\theta.$$

Hence the result.

EXAMPLE 8. *Show that the sum of the intercepts of the tangent to* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *upon the co-ordinate axes is constant.*

(P.U. 66A, 80A; B.U. 54A, 93H; Bhag. 64A, 66S; M.U. 73A, 80, 91)

SOLUTION : Let $P(\alpha, \beta)$ be any point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.



Then the equation of the tangent at (α, β) will be

$$y - \beta = \left(\frac{dy}{dx}\right)_{\substack{x=\alpha \\ y=\beta}}(x - \alpha) \qquad \ldots(1)$$

Now differentiating the given equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}} \cdot \frac{dy}{dx} = -\frac{1}{\sqrt{x}} \quad \therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

$\therefore$ at the point (α, β), $\frac{dy}{dx} = -\frac{\sqrt{\beta}}{\sqrt{\alpha}}$.

Hence from (1), the equation of the tangent will be

$$y - \beta = -\frac{\sqrt{\beta}}{\sqrt{\alpha}}(x - \alpha)$$

$$\Rightarrow \quad y\sqrt{\alpha} - \beta\sqrt{\alpha} = -\sqrt{\beta}x + \alpha\sqrt{\beta}$$

$$\Rightarrow \quad y\sqrt{\alpha} + \sqrt{\beta}x = \alpha\sqrt{\beta} + \beta\sqrt{\alpha}$$

$$= \sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha} + \sqrt{\beta}) = \sqrt{\alpha}\sqrt{\beta}\sqrt{a};$$

$\because$ (α, β) lies on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, therefore $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{a}$.

Thus the equation of the tangent at the point (α, β) becomes

$$y\sqrt{\alpha} + \sqrt{\beta}x = \sqrt{\alpha}\sqrt{\beta}\sqrt{a}. \qquad \ldots(2)$$

Now we shall find out the intercept on the x-axis and y-axis of the st. line (2).

Let the st. line (2) cuts the x-axis at A and the y-axis at B.

Then the y-co-ordinate of $A = 0$.

To find out the x-co-ordinate of A, we put $y = 0$ in (2). $\therefore$ $x = \sqrt{\alpha}\sqrt{a}$

i.e. the intercept OA on the x-axis $= \sqrt{\alpha}\sqrt{a}$.

Similarly to find out the intercept on the y-axis we put $x = 0$. Therefore putting $x = 0$ in (2), $y = \sqrt{\beta}\sqrt{a}$.

i.e. the intercept OB on the y-axis $= \sqrt{\beta}\sqrt{a}$.

$\therefore$ the intercept OA on the x-axis + the intercept OB on the y-axis

$$= \sqrt{\alpha}\sqrt{a} + \sqrt{\beta}\sqrt{a} = \sqrt{a}(\sqrt{\alpha} + \sqrt{\beta})$$

$$= \sqrt{a}\sqrt{a} = a \text{ (const.)}$$

Hence the result.

EXAMPLE 9. *If x_1, y_1 be the intercepts on the co-ordinate axes made by the tangent at any point (x, y) of the curve*

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1, \text{ show that } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

(P.U. 56A, 60A, 63S; M.U. 64

SOLUTION : [See the fig. of Ex. 8]

Given that $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

Therefore differentiating we get

$$\frac{2}{3}\left(\frac{x}{a}\right)^{\frac{2}{3}-1}\cdot\frac{1}{a}+\frac{2}{3}\left(\frac{y}{b}\right)^{\frac{2}{3}-1}\cdot\frac{1}{b}\cdot\frac{dy}{dx}=0.$$

$$\Rightarrow \quad \frac{1}{b}\cdot\left(\frac{y}{b}\right)^{-\frac{1}{3}}\frac{dy}{dx}=-\frac{1}{a}\left(\frac{x}{a}\right)^{-\frac{1}{3}}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{b}{a}\cdot\frac{(y/b)^{1/3}}{(x/a)^{1/3}}=-\frac{b}{a}\cdot\frac{y^{1/3}}{b^{1/3}}\times\frac{a^{1/3}}{x^{1/3}}$$

$$=-\frac{b^{2/3}}{a^{2/3}}\cdot\frac{y^{1/3}}{x^{1/3}}.$$

Therefore the equation of the tangent at the point (x, y) is

$$Y-y=-\frac{b^{2/3}}{a^{2/3}}\cdot\frac{y^{1/3}}{x^{1/3}}(X-x)$$

$$\Rightarrow \quad a^{2/3}x^{1/3}Y-a^{2/3}x^{1/3}y=-b^{2/3}y^{1/3}X+b^{2/3}y^{1/3}x$$

$$\Rightarrow \quad b^{2/3}y^{1/3}X+a^{2/3}x^{1/3}Y=b^{2/3}y^{1/3}x+a^{2/3}x^{1/3}y$$

$$=x^{1/3}y^{1/3}(b^{2/3}x^{2/3}+a^{2/3}y^{2/3}).$$

Dividing this by $a^{2/3}b^{2/3}$, we get

$$\frac{y^{1/3}X}{a^{2/3}}+\frac{x^{1/3}Y}{b^{2/3}}=x^{1/3}y^{1/3}\left(\frac{x^{2/3}}{a^{2/3}}+\frac{y^{2/3}}{b^{2/3}}\right)$$

$$=x^{1/3}y^{1/3} \quad \because \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1. \qquad ...(1)$$

We will have to put $Y = 0$ for the point at which this tangent (1) cuts the x-axis.

Hence putting $Y = 0$ in (1),

$$\frac{y^{1/3}X}{a^{2/3}}=x^{1/3}y^{1/3} \quad \therefore \quad X=a^{2/3}x^{1/3}.$$

i.e. the intercept on the X-axis $= a^{2/3}x^{1/3}$ i.e. $x_1 = a^{2/3}x^{1/3}$.

Similarly by putting $X = 0$, the intercept on the y-axis

i.e. $y_1 = b^{2/3}y^{1/3}$.

Thus $x_1 = a^{2/3}x^{1/3}$; $\quad y_1 = b^{2/3}y^{1/3}$,

$\therefore \quad x_1^2 = a^{4/3}x^{2/3}$; $\quad y_1^2 = b^{4/3}y^{2/3}$.

$$\therefore \quad \frac{x_1^2}{a^2}=\frac{x^{2/3}}{a^{2/3}}; \quad \frac{y_1^2}{b^2}=\frac{y^{2/3}}{b^{2/3}}$$

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1.$$

Note : We could have solved the problem in an easier way by taking the point $(a\cos^3\theta, b\sin^3\theta)$ on the given curve.

EXAMPLE 10. *Show that the portion of the tangent to the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ *which is intercepted between the axes is of constant length and find the area of the portion included between the axes and the tangent.*

(R.U. 62S, 63A, 68A, 71A; B.U. 56S, 58S, 65A; Bhag. 63A, 71A; M.U. 65S, 77A, 79; P.U. 33S, 46A, 48S, 49A, 51A, 55S, 70S, 75A)

SOLUTION : [See the fig. of Ex. 8]

First of all, we find out the equation of the tangent at any point (α, β). Given the equation of the curve is

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Differentiating w.r.t. x, we get

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0 \quad \Rightarrow \quad y^{-1/3}\frac{dy}{dx} = -x^{-1/3}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}.$$

$$\therefore \text{ at the point } (\alpha, \beta), \quad \frac{dy}{dx} = -\frac{\beta^{1/3}}{\alpha^{1/3}}.$$

Hence the equation of the tangent at the point (α, β) is

$$y - \beta = -\frac{\beta^{1/3}}{\alpha^{1/3}}(x - \alpha)$$

$$\Rightarrow \quad \alpha^{1/3}y - \alpha^{1/3}\beta = -\beta^{1/3}x + \alpha\beta^{1/3}$$

$$\Rightarrow \quad \beta^{1/3}x + \alpha^{1/3}y = \alpha\beta^{1/3} + \alpha^{1/3}\beta$$

$$= \alpha^{1/3}\beta^{1/3}(\alpha^{2/3} + \beta^{2/3}) = \alpha^{1/3}\beta^{1/3}.a^{2/3}. \quad \ldots(1)$$

Since (α, β) lies on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, therefore

$$\alpha^{2/3} + \beta^{2/3} = a^{2/3}.$$

Now to find out the intercept on the x-axis, we shall put $y = 0$ in (1).

Hence putting $y = 0$ in (1),

$$\beta^{1/3}x = \alpha^{1/3}\beta^{1/3}a^{2/3} \quad \Rightarrow \quad x = \alpha^{1/3}a^{2/3}.$$

$\therefore$ the intercept of the tangent on the x-axis $= \alpha^{1/3}a^{2/3}$.

That is, the x-co-ordinate of the point at which the tangent cuts the x-axis $= \alpha^{1/3}a^{2/3}$.

In the same way, by puttting $x = 0$, in (1) it can be shown that the y-co-ordinate of the point at which the tangent cuts the y-axis is $\beta^{1/3}a^{2/3}$.

Let this tangent cut the x-axis at A and the y-axis at B.

Thus the co-ordinates of $A = (\alpha^{1/3}a^{2/3}, 0)$ and the co-ordinates of $B = (0, \beta^{1/3}a^{2/3})$.

Now $AB^2 = (\alpha^{1/3}a^{2/3} - 0)^2 + (0 - \beta^{1/3}a^{2/3})^2 = \alpha^{2/3}a^{4/3} + \beta^{2/3}a^{4/3}$

$= a^{4/3}(\alpha^{2/3} + \beta^{2/3}) = a^{4/3}.a^{2/3} = a^2$.

$\therefore$ $AB = a$ (constant).

Thus the first result is proved.

Now, we have to find out the area of ΔOAB.

For this, we know that $\Delta OAB = \frac{1}{2}OA \times OB$

$$= \frac{1}{2} \cdot \alpha^{1/3}a^{2/3} \times \beta^{1/3}a^{2/3} = \frac{1}{2}\, a^{4/3}(\alpha\beta)^{1/3}.$$

EXAMPLE 11. *If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that*

$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1.$$

(P.U. 54H, 90H; B.U. 72A, 92H; M.U. 73H; Bhag. 91H, 93H)

SOLUTION : Here $x^3 + y^3 = a^3$.

Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}.$$

Therefore the equation of the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{x_1^2}{y_1^2}(x - x_1) \Rightarrow yy_1^2 - y_1^3 = -xx_1^2 + x_1^3$$

$\Rightarrow$ $xx_1^2 + yy_1^2 = x_1^3 + y_1^3 = a^3$; ...(1)

since (x_1, y_1) lies on the curve.

If the tangent passes through the point (x_2, y_2), then we shall have

$$x_2.x_1^2 + y_2y_1^2 = a^3 \quad ...(2)$$

Also $x_1^3 + y_1^3 = a^3$ ...(3)

and $x_2^3 + y_2^3 = a^3$ ...(4)

(2) – (4)

$\Rightarrow$ $x_2x_1^2 + y_2y_1^2 = x_1^3 + y_1^3$

$\Rightarrow$ $x_2x_1^2 - x_1^3 = y_1^3 - y_2y_1^2$

$\Rightarrow$ $x_1^2(x_2 - x_1) = y_1^2(y_1 - y_2)$ ...(5)

(1) – (4)

$\Rightarrow$ $x_2x_1^2 + y_2y_1^2 = x_2^3 + y_2^3$

$\Rightarrow$ $x_2x_1^2 - x_2^3 = y_2^3 - y_2y_1^2$

$\Rightarrow$ $x_2(x_1^2 - x_2^2) = y_2(y_2^2 - y_1^2)$ ...(6)

Now (5) ÷ (6)

$\Rightarrow$ $\dfrac{x_1^2(x_2 - x_1)}{x_2(x_1^2 - x_2^2)} = \dfrac{y_1^2(y_1 - y_2)}{y_2(y_2^2 - y_1^2)}$

$\Rightarrow$ $-\dfrac{x_2}{x_1^2}(x_2 + x_1) = -\dfrac{y_2}{y_1^2}(y_1 + y_2)$

$\Rightarrow$ $\dfrac{x_2^2}{x_1^2} + \dfrac{x_2}{x_1} = \dfrac{y_2}{y_1} + \dfrac{y_2^2}{y_1^2}$

$\Rightarrow$ $\dfrac{x_2}{x_1} - \dfrac{y_2}{y_1} = \dfrac{y_2^2}{y_1^2} - \dfrac{x_2^2}{x_1^2} = \left(\dfrac{y_2}{y_1} + \dfrac{x_2}{x_1}\right)\left(\dfrac{y_2}{y_1} - \dfrac{x_2}{x_1}\right)$

$\Rightarrow$ $\dfrac{x_2}{x_1} + \dfrac{y_2}{y_1} = -1.$

Hence the result

EXAMPLE 12. *If* $p = x\cos\alpha + y\sin\alpha$ *touches the curve* $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$, *prove that*

$$p^{\frac{m}{m-1}} = (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}}.$$

(B.U. 56A, 57S, 58A, 60A, 64S, 66A, 68H, 75A, 86; P.U. 31A, 37S, 38S, 55A, 66A, 73A, 86, 88; R.U. 73A, 76A; Mithila. 76A, 78A; M.U. 81, 90

SOLUTION : Given $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1.$

Differentiating w.r.t. x, we get

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}}$$

Hence the equation of the tangent at (x, y) is

$$Y - y = -\frac{b^m}{a^m} \cdot \frac{x^{m-1}}{y^{m-1}}(X - x)$$

$$\Rightarrow \quad (X - x)\frac{x^{m-1}}{a^m} + (Y - y)\frac{y^{m-1}}{b^m} = 0$$

$$\Rightarrow \quad \frac{Xx^{m-1}}{a^m} - \frac{x^m}{a^m} + \frac{Yy^{m-1}}{b^m} - \frac{y^m}{b^m} = 0$$

$$\Rightarrow \quad \frac{Xx^{m-1}}{a^m} + \frac{Yy^{m-1}}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \qquad \ldots(1)$$

But it is given then

$$x\cos\alpha + y\sin\alpha = p \qquad \ldots(2)$$

touches the given curve at the point (x, y). Therefore the two equations (1) and (2) represent the same equation, hence they are identical. Therefore, the coeffs. of x, y and the constant term will be proportional in both the equations. Hence

$$\frac{x^{m-1}}{a^m}\bigg|\cos\alpha = \frac{y^{m-1}}{b^m}\bigg|\sin\alpha = \frac{1}{p}$$

$$\Rightarrow \quad \frac{x^{m-1}}{a^m\cos\alpha} = \frac{y^{m-1}}{b^m\sin\alpha} = \frac{1}{p}$$

$$\therefore \quad x^{m-1} = \frac{a^m\cos\alpha}{p} \quad \Rightarrow \quad x = (a^m\cos\alpha)^{\frac{1}{m-1}}$$

and $$y^{m-1} = \frac{b^m\sin\alpha}{p} \quad \Rightarrow \quad y = \left(\frac{b^m\sin\alpha}{p}\right)^{\frac{1}{m-1}}$$

Now since (x, y) satisfies the equation $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$,

therefore

$$\frac{\left(\frac{a^m\cos\alpha}{p}\right)^{\frac{m}{m-1}}}{a^m} + \frac{\left(\frac{b^m\sin\alpha}{p}\right)^{\frac{m}{m-1}}}{b^m} = 1$$

$$\Rightarrow \quad \frac{a^{\frac{m^2}{m-1}}\cos^{\frac{m}{m-1}}\alpha}{a^m p^{\frac{m}{m-1}}} + \frac{b^{\frac{m^2}{m-1}}\sin^{\frac{m}{m-1}}\alpha}{b^m p^{\frac{m}{m-1}}} = 1$$

$$\Rightarrow \quad a^{\frac{m^2}{m-1}-m}\cos^{\frac{m}{m-1}}\alpha + b^{\frac{m^2}{m-1}-m}\sin^{\frac{m}{m-1}}\alpha = p^{\frac{m}{m-1}}$$

$$\Rightarrow \quad a^{\frac{m}{m-1}}\cos^{\frac{m}{m-1}}\alpha + b^{\frac{m}{m-1}}\sin^{\frac{m}{m-1}}\alpha = p^{\frac{m}{m-1}}$$

$$\therefore \quad (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}.$$

EXAMPLE 13. *Prove that the condition* $x\cos\alpha + y\sin\alpha = p$ *should touch* $x^m y^n = a^{m+n}$ *is*

$$p^{m+n}m^m n^n = (m+n)^{m+n}a^{m+n}\sin^n\alpha\cos^m\alpha.$$

(P.U. 54A, 60S, 61S, 62A, 64A, 69A, 76A, 95H; B.U. 58A, 93; M.U. 63A, 72S, 78A; Bhag. 63S, 68A, 72A, 86A, 89H, 94H; R.U. 72S, 81)

Prove that the portion of the tangent intercepted between the axes is divided at point of contact in a given ratio. **(Bhag, 89H, 94H)**

SOLUTION : Here $x^m y^n = a^{m+n}$.

Differentiating w.r.t. x, we get

$$x^m . ny^{n-1}\frac{dy}{dx} + y^n . mx^{m-1} = 0$$

$$\Rightarrow \quad nx^m y^{n-1}\frac{dy}{dx} = -mx^{m-1}y^n$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{m}{n}\cdot\frac{x^{m-1}y^n}{x^m y^{n-1}} = -\frac{m}{n}\cdot\frac{y}{x}.$$

Hence the equation of the tangent at $P(x, y)$ is

$$Y - y = -\frac{m}{n}\cdot\frac{y}{x}(X - x)$$

$$\Rightarrow \quad nxY - nxy = -myX + mxy$$

$$\Rightarrow \quad myX + nxY = (m+n)xy \qquad \ldots(1)$$

But it is given that

$$x\cos\alpha + y\sin\alpha = p \qquad \ldots(2)$$

$$\therefore \quad \frac{my}{\cos\alpha} = \frac{nx}{\sin\alpha} = \frac{(m+n)xy}{p}$$

From first and third, $x = \dfrac{pm}{(m+n)\cos\alpha}$

From second and third $y = \dfrac{pn}{(m+n)\sin\alpha}$

Now since (x, y) satisfies the equation $x^m y^n = a^{m+n}$, therefore

$$\frac{p^m m^m}{(m+n)^m \cos^m\alpha} \times \frac{p^n n^n}{(m+n)^n \sin^n\alpha} = a^{m+n}$$

$$\Rightarrow \quad p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \cos^m\alpha \sin^n\alpha.$$

Hence the result.

This proves the first part.

Let the tangent (1) meet the x-axis and y-axis in point A and B respectively. Then the co-ordinates of A are $\left(\frac{m+n}{m}x, 0\right)$ and those of B are $\left(0, \frac{m+n}{n}y\right)$.

Let the point $P(x, y)$ divide AB in the ratio of $p : q$. Then

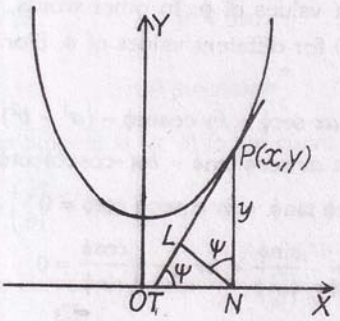
$$x = \frac{p \cdot \frac{m+n}{m}x + 0}{p+q} \quad \text{and} \quad y = \frac{0 + q \cdot \frac{m+n}{n}y}{p+q}$$

$$\Rightarrow \quad \frac{p}{p+q} = \frac{m}{m+n} \quad \text{and} \quad \frac{q}{p+q} = \frac{n}{m+n}.$$

Dividing, we get $\frac{p}{q} = \frac{m}{n}$. That is, the point divides the tangent intercepted between the axes in the constant ratio of $m : n$.

EXAMPLE 14. *Prove that for the catenary $y = c\cosh\frac{x}{c}$, the length of the perpendicular from the foot of the ordinate on the tangent is constant.*

(P.U. 47S, 51S; Bhag 62A; M.U. 66A; B.U. 68A; R.U. 67, 72H)



SOLUTION : Let $P(x, y)$ be any point on the curve. From P, draw PN perpendicular to x-axis. Then $PN = y$.

At P, draw a tangent PT which meets the x-axis at T and PT makes an angle ψ with the x-axis i.e. $\angle PTX = \psi$.

From N, NL has been drawn perpendicular to the tangent PT. We have to find the length of NL.

From the figure, $\angle TPN = 90° - \psi$.

$$\therefore \quad \sin(90° - \psi) = \frac{NL}{PN} \Rightarrow \cos\psi = \frac{NL}{y}$$

$$\therefore \quad NL = y\cos\psi \qquad \ldots(1)$$

Now from the equation of the given curve $y = c\cosh\frac{x}{c}$,

$$\frac{dy}{dx} = c\sinh\frac{x}{c}\cdot\frac{1}{c} = \sinh\frac{x}{c}$$

$$\Rightarrow \quad \tan\psi = \sinh\frac{x}{c} \quad \therefore \quad \cos\psi = \frac{1}{\sqrt{1+\sinh^2\frac{x}{c}}} = \frac{1}{\cosh\frac{x}{c}} = \frac{c}{y}.$$

$$\therefore \quad NL = y\cos\psi = y\cdot\frac{c}{y} = c \text{ (const.)}$$

Hence the result.

EXAMPLE 15. *Show that the normal to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *touches the curve* $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. **(P.U. 59)**

SOLUTION : We know from co-ordinate Geometry (or it can be derived as on page 191) that the equation of the normal to the given ellipse at the point 'ϕ' (i.e. $a\cos\phi$, $b\sin\phi$) is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2. \qquad \ldots(1)$$

We have to find the equation of the curve for which the equation (1) is the tangent for different values of ϕ. In other words, we have to find the envelope of the line (1) for different values of ϕ. [For envelope, see chapter 10.]

Let $\quad f(\phi) \equiv ax\sec\phi - by\,\text{cosec}\phi - (a^2 - b^2) = 0 \qquad \ldots(2)$

Then $\quad f'(\phi) \equiv ax\sec\phi.\tan\phi - by(-\text{cosec}\phi\cot\phi) = 0$

$$\Rightarrow \quad ax\sec\phi\tan\phi + by\,\text{cosec}\phi\cot\phi = 0$$

$$\Rightarrow \quad ax\frac{1}{\cos\phi}\cdot\frac{\sin\phi}{\cos\phi} + by\frac{1}{\sin\phi}\cdot\frac{\cos\phi}{\sin\phi} = 0$$

$$\Rightarrow \quad \frac{ax\sin\phi}{\cos^2\phi} + \frac{by\cos\phi}{\sin^2\phi} = 0$$

$$\Rightarrow \quad ax\sin^3\phi + by\cos^3\phi = 0$$

$$\Rightarrow \quad \frac{\sin^3\phi}{by} + \frac{\cos^3\phi}{ax} = 0$$

$$\Rightarrow \quad \frac{\sin^3\phi}{by} = -\frac{\cos^3\phi}{ax}$$

$$\Rightarrow \quad \frac{\sin\phi}{(by)^{1/3}} = \frac{\cos\phi}{(-ax)^{1/3}} = \frac{1}{\sqrt{(by)^{2/3} + (ax)^{2/3}}}$$

$$\Rightarrow \quad \sin\phi = \frac{(by)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}} \text{ and } \cos\phi = \frac{-(ax)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}} \qquad ...(3)$$

Now, eliminating ϕ between (2) and (3), we get

$$ax\frac{\sqrt{(ax)^{2/3} + (by)^{2/3}}}{-(ax)^{1/3}} - by \cdot \frac{\sqrt{(ax)^{2/3} + (by)^{2/3}}}{(by)^{1/3}} = a^2 - b^2$$

$$\Rightarrow \quad \sqrt{(ax)^{2/3} + (by)^{2/3}}\ \{(ax)^{2/3} + (by)^{2/3}\} = a^2 - b^2$$

by suitably taking the sign of the square root

$$\Rightarrow \quad \{(ax)^{2/3} + (by)^{2/3}\}^{3/2} = a^2 - b^2$$

$$\therefore \quad (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

Hence the result.

Example 13

1. Find the equation of the tangents and normals at the point (x_1, y_1) on each of the following curves:

(i) $x^2 + y^2 = a^2$ (ii) $y^2 = 4ax$ (iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iv) $xy = c$ (v) $y = c\cos h\frac{x}{c}$

2. Show that the tangent at (a, b) to the curve

$$\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2 \text{ is } \frac{x}{a} + \frac{y}{b} = 2. \qquad \textbf{(R.U. 64A, 83)}$$

3. Show that the curves represented by $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ for different values of n, have a common tangent at the point (a, b). Find the equation of the common tangent. **(P.U. 41S, 45A; B.U. 61A; M.U. 84)**

4. Find the equation to the tangent at the point (x_1, y_1) to the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1.$$

5. Find the equation of the tangent at 't' on the curve

$$x = a\sin^3 t, \quad y = b\cos^3 t.$$

6. Find the equation of the normal at any point on the curve $y^m = ax^n$.

7. Find the equation of the tangent at a point 't' on the curve

$$x = a(t + \sin t), \quad y = a(1 - \cos t). \qquad \textbf{(P.U. 70A)}$$

8. *Find the tangents at the origin to the curves*

(i) $y^2 = 4ax$ (ii) $x^3 + y^3 = 3axy$ (iii) $x^2y^2 = a^2(x^2 - y^2)$

9. Find where the tangent is parallel to the axis of x for the following curves—

(i) $y = (x - 1)(x - 2)(x - 3)$ (ii) $x^3 + y^3 = 3axy$ (iii) $x^3 + y^3 = a^3$

(iv) $ax^2 + 2hxy + by^2 = 1$ (v) $3b^2y = x^3 - 3ax^2$

10. Find at what points on the curve $\frac{x^3}{a} + \frac{y^3}{b} = xy$ the tangent is parallel to one of the co-ordinate axes. **(P.U. 50S; B.U. 69S)**

11. Find the points on the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ where the tangent is parallel to $y = 2x$.

12. Find the points on the curve $y = x^2 + 3x + 4$, tangents at which pass through the origin. **(P.U. 65A)**

13. Find the equations of the tangent and normal to the curve $y(x - 2)(x - 3) - x + 7 = 0$ at the point where it cuts the axis of x.

14. Show that in the curve $y = (x - 1)(x - 2)(x - 3)$ the tangents at the first and third intersections with the x-axis are parallel and at the middle intersection the tangent makes an angle of 135° with that axis.

15. Show that the points of contact of tangents from (α, β) to $x^m y^n = a^{m+n}$ are situated on the hyperbola. **(P.U. 37A)**

16. If x_1, y_1 be the part of the axes of x and y intercepted by the tangent at any point on the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1, \text{ show that } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

(P.U. 56A, 60A, 63A; M.U. 64A)

17. Show that the part of the tangent to $xy = c^2$ included between the co-ordinate axes is bisected at the point of tangency.

(B.U. 55A; Bhag. 65A, 68A)

18. Find the equation of the tangent to the curve $x = at^2$, $y = at^3$ at any point 't'. Find where the tangent meets **(P.U. 57A)**

19. Find the equation of the tangent to $x^3 = ay^2$ at $(4am^2, 8am^3)$ and also the point in which the tangent cuts the curve again.

20. Find the equation of the tangent at any point 'θ' to the curve $x = a\cos 2\theta$, $y = 2\sqrt{2}\,a\sin\theta$; and if any two tangents are at right angles, prove that they must intersect on the line $x = 2a$. **(P.U. 53A)**

21. Find the equation of the tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at the point $(a\cos^3\theta, a\sin^3\theta)$. If P and Q be the points corresponding to $\theta, \frac{\pi}{2} + \theta$ and the tangents at these points meet at T, prove that $OP^2 + OQ^2 + 3OT^2 = \text{const.}$ **(P.U. 57S)**

22. Find the abscissa of the point on the curve $ay^2 = x^3$ the normal at which cuts off equal intercepts from the co-ordinate axes. **(P.U. 61; B.U. 58H, 67H)**

23. Find where the normal to $\sqrt{xy} = a + x$ makes equal intercepts on the axes. **(P.U. 38A; B.U. 61S)**

24. If $px + qy = 1$ touches the curve $(ax)^n + (by)^n = 1$, show that

$$\left(\frac{p}{a}\right)^{\frac{n}{n-1}} + \left(\frac{q}{b}\right)^{\frac{n}{n-1}} = 1.$$

25. If $x\cos\alpha + y\sin\alpha = p$ touches the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$, prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$. **(M.U. 87; B.U. 90H; Bhag. 91)**

26. Find the condition that $x\cos\theta + y\sin\theta = p$ is normal to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \textbf{(R.U. 62S)}$$

27. Show that the normal to the parabola $y^2 = 4ax$ touches the curve

$$27ay^2 = 4(x - 2a)^3. \qquad \textbf{(P.U. 63A; B.U. 66S)}$$

28. Show that the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the curve

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}. \qquad \textbf{(P.U. 59A)}$$

29. The normal to the parabola $y^2 = 4ax$ at $(am^2, 2am)$ cuts the axis in P. Prove that the line through P at right angles to the normal touches the parabola $y^2 + 4a(x - 2a) = 0$. **(P.U. 58A; M.U. 65A)**

30. Prove that the locus of intersection of perpendicular tangents to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 = \frac{a^2}{2}\cos^2 2\theta$. **(P.U. 55H)**

[Hints: Let $P(\alpha)$ and $Q(\beta)$ be two points on the curve. Then the equation of the tangent at P and Q are

$$x\sin\alpha + y\cos\alpha = a\sin\alpha\cos\alpha \quad ...(1)$$

$$x\sin\beta + y\cos\beta = a\sin\beta\cos\beta \quad ...(2)$$

Since the tangents (1) are (2) are perpendicular, therefore $\sin\alpha\sin\beta + \cos\alpha\cos\beta = 0 \Rightarrow \cos(\alpha - \beta) = 0 \Rightarrow \alpha - \beta = \frac{\pi}{2}$ i.e. $\alpha = \beta + \frac{\pi}{2}$.

With this, (1) reduces to

$$x\cos\beta - y\sin\beta = -a\cos\beta\sin\beta \quad ...(3)$$

Eliminating β from (2) and (3), we get

$$\frac{\sin\beta}{y+x} = \frac{\cos\beta}{y-x} = \frac{a\sin\beta\cos\beta}{x^2+y^2}$$

$$\Rightarrow \quad a\cos\beta = \frac{x^2+y^2}{y+x} \text{ and } a\sin\beta = \frac{x^2+y^2}{y-x}.$$

Hence squaring and adding,

$$(x^2+y^2)^3 = \frac{a^2}{2}(x^2-y^2)^2 \text{ etc.}]$$

ANSWERS

1. (i) $xx_1 + yy_1 = a^2$; $\frac{x}{x_1} = \frac{y}{y_1}$ (ii) $yy_1 = 2a(x + x_1)$; $\frac{x - x_1}{2a} + \frac{y - y_1}{y_1} = 0$

(iii) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$; $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$

(iv) $\frac{x}{x_1} + \frac{y}{y_1} = 2$; $(x - x_1)x_1 = (y - y_1)y_1$

(v) $y - y_1 = \sin h\frac{x_1}{c}(x + x_1)$; $y - y_1 + \frac{x - x_1}{\sin h\frac{x_1}{c}} = 0$

3. $\frac{x}{a} + \frac{y}{b} = 2$ **4.** $\frac{xx_1^{m-1}}{a^m} + \frac{yy_1^{m-1}}{b^m} = 1$. **5.** $\frac{x}{a\sin t} + \frac{y}{b\cos t} = 1$

6. $mx(X - x) + ny(Y - y) = 0$ **7.** $y = (x - at)\tan\frac{t}{2}$

8. (i) $x = 0$ (ii) $x = 0$, $y = 0$ (iii) $y = \pm x$

9. (i) at points where $x = \pm \frac{1}{\sqrt{3}}$ (ii) $\{(2a)^{1/3}, (4a)^{1/3}\}$: (iii) $(0, a)$

(iv) where $ax + by = 0$ intersects the curve (v) $\left(2a, \frac{-4a^3}{3b^2}\right)$

10. The tangent is parallel to y-axis at the origin and also at

$$\left(\frac{1}{3} \cdot 2^{2/3} a^{2/3} b^{1/3}, \frac{1}{3} \cdot 2^{1/3} a^{1/3} b^{2/3}\right);$$

the tangent is parallel to x-axis at the origin and also at

$$\left(\frac{1}{3} \cdot 2^{1/3} a^{2/3} b^{1/3}, \frac{1}{3} \cdot 2^{2/3} a^{1/3} b^{2/3}\right)$$

11. $(1, 3), (2, 5), \left(\frac{3}{2}, \frac{15}{16}\right)$ **12.** $(2, 14), (-2, 2)$

13. $20y - x + 7 = 0, \quad 20x + y - 140 = 0$

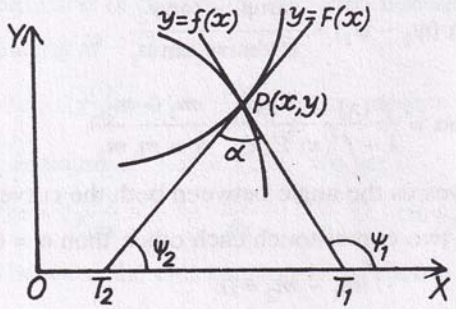
18. $2y = 3tx - at^3; \left(\frac{1}{4}at^2, \frac{1}{8}at^2\right)$ **19.** $y = 3mx - 4am^3; \ (am^2, am^3)$

20. $\sqrt{2}y \sin\theta + x = a(1 + 2\sin^2\theta)$ **21.** $x \sin\theta + y \cos\theta = a \sin\theta\cos\theta$

22. $x = \frac{4a}{9}$ **23.** At the points where $x = \pm \frac{a}{\sqrt{2}}$

26. $p = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$.

4.4 Angle of intersection of two curves



Let the equations of both the curves be

$$y = f(x) \text{ and } y = F(x).$$

Let the two curves cut each other at the point $P(x,y)$.

From P, draw the tangent PT_1 to the curve $y = f(x)$ and the tangent PT_2 to the curve $y = F(x)$ which meet the x-axis respectively at the points T_1 and T_2.

We know that at the point of intersection P, the angle between the two curves is the same as the angle between their tangents PT_1 and PT_2 at the point P.

That is, we want to find $\angle T_2PT_1$.

Let the tangent PT_1 make an angle ψ_1 with the x-axis and PT_2 make an angle ψ_2 with the x-axis;

i.e. $\angle PT_1X = \psi_1$ and $\angle PT_2X = \psi_2$.

Then $\angle T_2PT_1 = \psi_1 - \psi_2 = \alpha$.

Since PT_1 make an angle ψ_1 with the x-axis, therefore using the formula $\tan\psi = \dfrac{dy}{dx}$ for the curve $y = f(x)$, we get

$$\tan \psi_1 = \frac{dy}{dx} = f'(x) = m_1 \text{ (say)}$$

Again for the curve $y = F(x)$,

$$\tan \psi_2 = F'(x) = m_2 \text{ (say)}$$

$$\therefore \quad \tan(\psi_1 - \psi_2) = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}$$

$$\Rightarrow \quad \tan\alpha = \frac{f'(x) - F'(x)}{1 + f'(x)\,F(x)} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

This formula gives us the angle between both the curves.

Note : (i) If the two curves touch each other, then $\alpha = 0$.

$$\therefore \quad m_1 - m_2 = 0.$$

$$\Rightarrow \quad f'(x) - F'(x) = 0$$

i. e. $\quad \boldsymbol{f'(x) = F'(x).}$

(ii) If the two curves cut each other at right angle, then

$\alpha = 90, \quad \therefore \; 1 + m_1 m_2 = 0.$

i.e. $\boldsymbol{1 + f'(x)F'(x) = 0.}$

$$\Rightarrow \quad 1 + \frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \times \frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = 0; \quad \text{putting the values of } f'(x) \text{ and } F'(x)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} \cdot \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial F}{\partial y} = 0.$$

Worked out Examples

EXAMPLE 1. *Find the angle of intersection of the curves*

$$x^2 - y^2 = a^2,\ x^2 + y^2 = \sqrt{2}\,a^2. \qquad \textbf{(P.U. 40 A, 44S; B.U. 69S)}$$

SOLUTION : We know that the angle between the two curves is the same as the angle between their tangents at the point of intersection of the two curves . Hence we shall have to find the co-ordinates of their point of intersection by solving the equations of the two curves. Then we shall draw tangents separately to the two curves at that point. We, first of all, determine the point of intersection of these two curves. Given that

$$x^2 - y^2 = a^2 \text{ and } x^2 + y^2 = \sqrt{2}\,a^2.$$

$$\therefore \text{ Adding } 2x^2 = (\sqrt{2} + 1)a^2 \Rightarrow x^2 = \frac{\sqrt{2} + 1}{2}a^2$$

$$\text{Subtracting } 2y^2 = (\sqrt{2} - 1)a^2 \Rightarrow y^2 = \frac{\sqrt{2} - 1}{2}a^2$$

$$\therefore \qquad x = \frac{a}{\sqrt{2}}(\sqrt{2} + 1)^{1/2} \text{ and } y = \frac{a}{\sqrt{2}}(\sqrt{2} - 1)^{1/2} \qquad \ldots(1)$$

Thus the co-ordinates of their point of intersection are found out.

Now differentiating $x^2 - y^2 = a^2$, we get

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y} \quad \therefore \quad \tan\psi_1 = \frac{x}{y}.$$

Similarly differentiating $x^2 + y^2 = \sqrt{2}\,a^2$, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \quad \therefore \quad \tan\psi_2 = -\frac{x}{y}.$$

Let the angle between the two tangents be α, then $\alpha = \psi_2 - \psi_1$.

$$\therefore \qquad \tan\alpha = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2 \tan\psi_1}$$

$$= \frac{-\dfrac{x}{y} - \dfrac{x}{y}}{1 - \dfrac{x}{y} \cdot \dfrac{x}{y}} = \frac{-\dfrac{2x}{y}}{1 - \dfrac{x^2}{y^2}} = \frac{2xy}{x^2 - y^2}.$$

Now from (1), $xy = \frac{a}{\sqrt{2}}(\sqrt{2}+1)^{1/2} \times \frac{a}{\sqrt{2}}(\sqrt{2}-1)^{1/2}$

$$= \frac{a^2}{2}\{(\sqrt{2}+1)(\sqrt{2}-1)\}^{1/2} = \frac{a^2}{2}.$$

$$x^2 - y^2 = \frac{a^2}{2}\{(\sqrt{2}+1) - (\sqrt{2}-1)\} = a^2.$$

$\therefore \tan\alpha = \frac{a^2}{a^2} = 1 \quad \therefore \alpha = 45°.$

EXAMPLE 2. *Find the condition that the conics* $ax^2 + by^2 = 1$, $a_1x^2 + b_1y^2 = 1$ *shall cut orthogonally.*

(Bhag. 63A, 90, 94; B.U. 55S, 59A; M.U. 66A; 68S, 74A, 92; P.U. 54S, 61A, 67A, 72A)

SOLUTION : If the two curves cut each other orthogonally, then at their point of intersection the angle between their tangents will be one right angle (= 90°).

Given $ax^2 + by^2 = 1$ ...(1)

$a_1x^2 + b_1y^2 = 1$ ...(2)

We know that if the two curves cut each other orthogonally, then the condition for this is

$1 + m_1m_2 = 0$ ...(3)

From (1), $a.2x + b.2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by} = m_1$

Similarly from (2), $\frac{dy}{dx} = -\frac{a_1x}{b_1y} = m_2.$

$\therefore$ From (3), $1 + \left(\frac{-ax}{by}\right)\left(\frac{-a_1x}{b_1y}\right) = 0$

$\Rightarrow \quad aa_1x^2 + bb_1y^2 = 0$ ...(4)

Here x and y are the co-ordinates of the point of intersection of the two curves. Therefore to find out the values of x and y we shall have to solve the equations of both the curves

$ax^2 + by^2 - 1 = 0$ ...(5)

$a_1x^2 + b_1y^2 - 1 = 0$ ...(6)

Subtracting (5) from (6),

$(a - a_1)x^2 + (b - b_1)y^2 = 0$

$\Rightarrow (a - a_1)x^2 = -(b - b_1)y^2$

But from (4), $aa_1x^2 = -bb_1y^2.$

$\therefore$ Dividing $\dfrac{a-a_1}{aa_1} = \dfrac{b-b_1}{bb_1}$

$\Rightarrow \quad \dfrac{1}{a_1} - \dfrac{1}{a} = \dfrac{1}{b_1} - \dfrac{1}{b}$

$\therefore \quad \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a_1} - \dfrac{1}{b_1}$

This is the required condition.

Example 14

1. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$ cut orthogonally.
2. Find the angles of intersection of the following curves:
   (i) $x^2 - y^2 = 2a^2$ and $x^2 + y^2 = 4a^2$. **(B.U. 69S)**
   (ii) $y = x^3$ and $6y = 7 - x^2$. (iii) $y = x^2$ and $6y = 7 - x^3$.
   (iv) $y^2 = 4ax$ and $x^2 = 4by$. **(M.U. 70A)**
   (v) $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$. **(P.U. 64)**
3. Find the condition that the curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ should intersect orthogonally. **(Bhag. 92)**

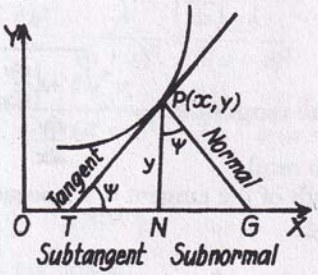
Answers

2. (i) $\dfrac{\pi}{3}$ (ii) $\dfrac{\pi}{2}$ (iii) $\dfrac{\pi}{2}$

   (iv) $\dfrac{\pi}{2}$ at (0, 0) and $\tan^{-1}\dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$ at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ (v) $\dfrac{\pi}{4}$

3. $a - b = a' - b'$.

5.5. Length of Normal, Subnormal, tangents and Subtangents.



Let the equation of the curve be $y = f(x)$. Let P be any point on the curve whose co-ordinates are (x, y).

From P, draw PN perp. to x-axis. Then $ON = x$ and we draw a tangent PT and normal PG at the point P of the curve which meets the x-axis at T and G respectively.

Then the length TN (i.e. the projection of the tangent PT on the x-axis) is called the **subtangent** and NG (i.e. the projection of the normal PG on the x-axis) is called the **subnormal.**

Here we want to find the lengths of the subtangent TN, subnormal NG, tangent PT and normal PG.

Let the tangent PT make an angle ψ with the x – axis. $\therefore \angle NPG = \psi$.

(i) Now from ΔTNP

$$\tan\Psi = \frac{y}{TN} \quad \therefore \quad TN = y/\tan\psi = \frac{y}{\frac{dy}{dx}}.$$

Hence **the length of the subtangent** $= \dfrac{y}{\frac{dy}{dx}}$.

(ii) Again, from ΔPNG,

$$\tan\psi = \frac{NG}{y} \quad \therefore \quad NG = y\tan\psi = y\frac{dy}{dx}.$$

Hence **the length of the subnormal** $= y\dfrac{dy}{dx}$.

(iii) $PT^2 = PN^2 + TN^2 = y^2 + \dfrac{y^2}{\left(\frac{dy}{dx}\right)^2} = y^2\left[1 + \dfrac{1}{\left(\frac{dy}{dx}\right)^2}\right]$

$$= y^2\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\left(\frac{dy}{dx}\right)^2}\right]$$

$\therefore$ **The length of the tangent** $PT = \dfrac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$.

Note : Also, the length of the tangent $= y\,\text{cosec}\,\psi$.

(iv) $PG^2 = PN^2 + NG^2$

$$= y^2 + y^2\left(\frac{dy}{dx}\right)^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$\therefore$ **The length of the normal** $PG = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Note : The length of the normal is also $= PG = y \sec\psi$.

Worked out Examples

EXAMPLE 1. *Show that in the curve* $y = be^{-a/x}$, *the subtangent varies as the square of the abscissa.* **(B.U. 54S, 57A)**

SOLUTION : Differentiating the given curve $y = be^{-a/x}$, we get

$$\frac{dy}{dx} = be^{-a/x} \times \frac{a}{x^2} = ab\, e^{-a/x} \cdot \frac{1}{x^2}.$$

Now, from the formula, the length of the subtangent $= \dfrac{y}{\dfrac{dy}{dx}}$

$$= \frac{y}{abe^{-a/x}} \times x^2 = \frac{y}{ay} \cdot x^2; \text{from the equation of the curve} = \frac{1}{a}x^2$$

$\therefore$ subtangent $\propto x^2$.

EXAMPLE 2. *Show that in the curve* $by^2 = (a+x)^3$, *the square of the subtangent varies as the subnormal.* **(P.U. 32S; B.U. 60A, 64A; Mithila. 76A; R.U. 80)**

SOLUTION : Differentiating the given curve $by^2 = (a+x)^3$, we get

$$b \cdot 2y\frac{dy}{dx} = 3(a+x)^2 \quad \therefore \quad \frac{dy}{dx} = \frac{3(a+x)^2}{2by}.$$

Now, from the formula, the length of the subtangent $= \dfrac{y}{\dfrac{dy}{dx}}$

$$= \frac{y \times 2by}{3(a+x)^2} = \frac{2b}{3} \cdot \frac{y^2}{(a+x)^2},$$

and the length of subnormal $= y\dfrac{dy}{dx} = y \cdot \dfrac{3(a+x)^2}{2by} = \dfrac{3(a+x)^2}{2b}$.

$$\therefore \quad \frac{(\text{subtangent})^2}{\text{subnormal}} = \left\{\frac{2b}{3} \cdot \frac{y^2}{(a+x)^2}\right\}^2 \times \frac{2b}{3(a+x)^2}.$$

$$= \frac{4b^2}{9} \cdot \frac{y^4}{(a+x)^4} \times \frac{2b}{3(a+x)^2}$$

$$= \frac{8b^3}{27} \cdot \frac{y^4}{(a+x)^6} = \frac{8b^3}{27} \cdot \frac{(a+x)^6}{b^2(a+x)^6}; \text{ from the equation of the curve}$$

$$= \frac{8b}{27} = \text{const.}$$

$\therefore$ (subtangent)$^2 \propto$ subnormal.

EXAMPLE 3. *In the catenary* $y = a\cos h\frac{x}{a}$, *prove that the length of the portion of the normal intercepted between the curve and the x-axis varies as* y^2.

(P.U. 44A, 50S, 64S; B.U. 56S, 58S, 59S, 62S; R.U. 62A; Bhag. 65A, 69A; M.U. 70A, 91H)

SOLUTION : In this problem, we have to find out the length of the normal.

From the formula the length of the normal $= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ ...(1)

Given the equation of the curve $y = a\cos h\frac{x}{a}$.

$\therefore \quad \frac{dy}{dx} = a\sin h\frac{x}{a}\cdot\frac{1}{a} = \sin h\frac{x}{a}$.

$\therefore$ from (1) the length of the normal $= y\sqrt{1 + \sin h^2\frac{x}{a}}$

$= y\cdot\sqrt{\cos h^2\frac{x}{a}} = y\cos h\frac{x}{a} = y\cdot\frac{y}{a}$; from the eqn. of the curve

$= \frac{1}{a}y^2 \quad \therefore$ Normal $\propto y^2$.

Example 16

1. Show that at any point of the parabola $y^2 = 4ax$, the subnormal is constant and the subtangent varies as the abscissa of the point of contact.
2. Show that in the curve $y = be^{x/a}$, the sub-tangent is constant and sub-normal is $\frac{y^2}{a}$. **(P.U. 37A; Bhag. 63S, 69S; B.U. 66S; M.U. 67A)**
3. Show that at any point of the curve $x^m y^n = a^{m+n}$, the subtangent varies as the abscissa of the point. **(B.U. 53S; R.U. 66A)**
4. Show that in any curve, the rectangle contained by the subtangent and the sub-normal is equal to the square of the corresponding ordinate. **(P.U. 65S; Bhag. 85)**
5. Show that in any curve (length of tangent)^2. subnormal = (length of normal)^2. subtangent. **(R.U. 64A)**
6. In the curve $x^{m+n} = a^{m+n}y^{2n}$, prove that the mth power of the sub-tangent varies as the nth power of the subnormal. **(P.U. 45S; B.U. 60A; R.U. 69A)**
7. Show that in the curve $y = a\log(x^2 - a^2)$ the sum of the tangent and subtangent varies as the product of the co-ordinates of the point. **(P.U. 47A, 53S, 63A; M.U. 71A; B.U. 67A)**

8. Prove that in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent.

9. Show that the length of the tangent at any point on the following curves is constant:

(i) $x = b(\cos t + \log \tan \frac{1}{2} t)$, $y = a \sin t$

(ii) $x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$ **(P.U. 62S)**

(Put $y = a \sin t$)

10. Find the lengths of the subtangent, subnormal, tangent and normal of the following curves:

(i) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at 'θ'.

(ii) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ at

ANSWERS

10. (i) $a \sin\theta$, $2a \sin^2\frac{\theta}{2} \tan\frac{\theta}{2}$, $2a \sin\frac{\theta}{2}$, $2a \sin\frac{\theta}{2}\tan\frac{\theta}{2}$

(ii) $y \cot t$, $y \tan t$, $y \operatorname{cosec} t$, $y \sec t$.

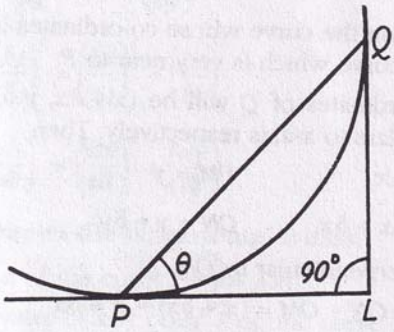
5.6. An Important limit

Let P and Q be any two points on the curve such that the arc PQ is at every stage concave w.r.t. the chord PQ. We want to show that

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1.$$

We suppose that the point Q is very near to the point P.

From P, we draw PL tangent to the curve. Again, from Q, we draw QL perpendicular to PL.



Let $\angle LPQ = \theta$.

Then from the figure,

chord $PQ <$ arc $PQ <$ $PL + LQ$

$$\Rightarrow \quad 1 < \frac{\text{arc } PQ}{\text{chord } PQ} < \frac{PL}{PQ} + \frac{LQ}{PQ}$$

$$\Rightarrow \quad 1 < \frac{\text{arc PQ}}{\text{chord}\,PQ} < \cos\theta + \sin\theta.$$

Now, let the point Q approach P along the curve. In this case, the chord PQ, becomes the tangent at the point P and $\theta \to 0$.

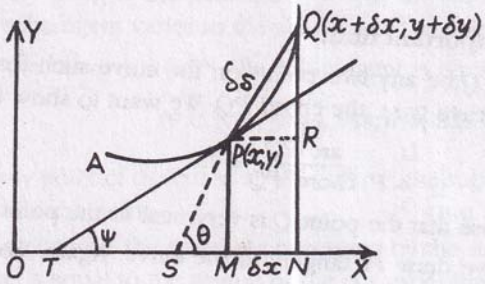
$$\therefore \quad 1 < \lim_{Q \to P} \frac{\text{arc PQ}}{\text{chord } PQ} < \lim_{\theta \to 0} (\cos\theta + \sin\theta)$$

$$\Rightarrow \quad 1 < \lim_{Q \to P} \frac{\text{arc PQ}}{\text{chord } PQ} < 1.$$

It follows, therefore, that

$$\lim_{Q \to P} \frac{\text{arc PQ}}{\text{chord}\,PQ} = 1.$$

5.7. Derivatives of arc—length in cartesian co-ordinates



Let P be any point on the curve whose co-ordinates are (x, y). Let Q be any other point on the curve which is very near to P.

Therefore the co-ordinates of Q will be $(x + \delta x, y + \delta y)$. From P draw PM and QN perpendiculars to x-axis respectively. Then

$$OM = x, \qquad PM = y$$

$$ON = x + \delta x, \qquad QN = y + \delta y.$$

From P draw PR perpendicular to QN.

$$\therefore \quad MN = ON - OM = (x + \delta x) - x = \delta x$$

$$\therefore \quad PR = MN = \delta x.$$

Again, $QR = QN - RN = QN - PM$

$= (y + \delta y) - y = \delta y.$

Let A be any point on the curve from where arc $AP = s$.

$\therefore$ arc $AQ = s + \delta s$. Hence arc $PQ = \delta s$.

Now from rt $\angle$ed ΔPRQ,

$$PQ^2 = PR^2 + RQ^2 = (\delta x)^2 + (\delta y)^2. \qquad ...(1)$$

But we know that when $Q \to P$, then $\dfrac{\text{chord } PQ}{\text{arc } PQ} \to 1$.

$\therefore$ In the limiting position, we can write arc PQ instead of chord PQ.

$\therefore$ from (1), $\delta s^2 = \delta x^2 + \delta y^2$.

Hence (i) $\left(\dfrac{\delta s}{\delta x}\right)^2 = 1 + \left(\dfrac{\delta y}{\delta x}\right)^2$.

$\therefore$ taking the limit,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \boldsymbol{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

(ii) Again $\left(\dfrac{\delta s}{\delta y}\right)^2 = 1 + \left(\dfrac{\delta x}{\delta y}\right)^2$.

$\therefore$ taking the limit,

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$$

$$\Rightarrow \quad \boldsymbol{\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}}.$$

(iii) Again $\left(\dfrac{\delta s}{\delta t}\right)^2 = \left(\dfrac{\delta x}{\delta t}\right)^2 + \left(\dfrac{\delta y}{\delta t}\right)^2$

$$\therefore \quad \boldsymbol{\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

5.8. Angle between the tangent and x-axis.

Let the equation of the curve be $y = f(x)$.

Let the co-ordinates of any point P on the curve be (x, y).

From P, a tangent PT has been drawn which meets OX in point T. Let the

angle between PT and x-axis be ψ. i.e. $\angle PTX = \psi$.

[See the fig of last Art]

Produce QP, which meets the x-axis at the point S.

Let $\angle PSX = \theta \quad \therefore \quad \angle QPR = \theta$.

Now $$\sin QPR = \frac{QR}{PQ} \Rightarrow \sin\theta = \frac{\delta y}{PQ} \qquad ...(1)$$

Similarly $$\cos\theta = \frac{\delta x}{PQ} \qquad ....(2)$$

$$\tan\theta = \frac{\delta y}{\delta x} \qquad ...(3)$$

Now let us suppose that $Q \to P$. Then in this case, the chord PQ, becomes the tangent at P.

Hence $\theta \to \psi$; also $\delta x \to 0$ and $\dfrac{\text{chord } PQ}{\text{arc } PQ} \to 1 \quad \therefore$ chord $PQ \to \delta s$.

Hence in the limiting position when $Q \to P$, from the equations (1), (2), (3), we get

(i) $\boldsymbol{\sin\psi = \frac{dy}{ds}}$

(ii) $\boldsymbol{\cos\psi = \frac{dx}{ds}}$

(iii) $\boldsymbol{\tan\psi = \frac{dy}{dx}}$

These formulae are very useful and should always be remembered.

Note : (i) If the tangent is parallel to x-axis, then $\psi = 0, \quad \therefore \quad \dfrac{dy}{dx} = 0$.

(ii) If the tangent is parallel to y-axis, then $\psi = 90°$.

$$\therefore \quad \frac{dy}{dx} = \infty \quad \therefore \quad \frac{dx}{dy} = 0.$$

EXAMPLE 1. *Find* $\dfrac{ds}{dx}$ *for the curves*

(i) $y^2 = 4ax$. **(R.U. 65S)** *(ii)* $x^{2/3} + y^{2/3} = a^{2/3}$.

SOLUTION : (i) Given $y^2 = 4ax$.

$\therefore$ Differentiating $2y\dfrac{dy}{dx} = 4a \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{y}$.

Now, $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4a^2}{y^2}$$

$$= \frac{y^2 + 4a^2}{y^2} = \frac{4ax + 4a^2}{4ax}; \quad \because y^2 = 4ax$$

$$= \frac{x + a}{x}.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\frac{x + a}{x}}.$$

(ii) Given $x^{2/3} + y^{2/3} = a^{2/3}$.

$\therefore$ Differentiating, we get

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0.$$

$$\Rightarrow \quad \frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0 \Rightarrow \frac{1}{y^{1/3}}\frac{dy}{dx} = -\frac{1}{x^{1/3}}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}.$$

$$\text{Now } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}}$$

$$= \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}; \text{ from the eqn. of the curve}$$

$$\therefore \quad \frac{ds}{dx} = \frac{a^{1/3}}{x^{1/3}} = \left(\frac{a}{x}\right)^{1/3}$$

EXAMPLE 2. *In the curve* $y = a \log \sec\frac{x}{a}$, *find* $\frac{ds}{dx}$ *and prove that* $x = a\psi$.

SOLUTION : Given the equation of the curve $y = a \log \sec \frac{x}{a}$.

Differentiating this, we get

$$\frac{dy}{dx} = a \cdot \frac{1}{\sec\frac{x}{a}} \cdot \sec\frac{x}{a}\tan\frac{x}{a} \cdot \frac{1}{a} = \tan\frac{x}{a}. \qquad \ldots(1)$$

$$\text{Now } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2\frac{x}{a} = \sec^2\frac{x}{a}$$

$$\therefore \quad \frac{ds}{dx} = \sec\frac{x}{a}.$$

Again, from (1), $\because \frac{dy}{dx} = \tan\frac{x}{a}$,

$$\therefore \tan\psi = \tan\frac{x}{a} \Rightarrow \psi = \frac{x}{a} \quad \therefore \quad x = a\psi.$$

Note : $\frac{ds}{dx} = \sec\frac{x}{a} \quad \therefore \frac{dx}{ds} = \cos\frac{x}{a}$.

Differentiating this w.r.t. s, we get

$$\frac{d^2x}{ds^2} = -\sin\frac{x}{a}\cdot\frac{1}{a}\cdot\frac{dx}{ds} = -\frac{1}{a}\sin\frac{x}{a}\cdot\cos\frac{x}{a}$$

$$= -\frac{1}{2a}\cdot 2\sin\frac{x}{a}\cos\frac{x}{a} = -\frac{1}{2a}\sin\frac{2x}{a}.$$

EXAMPLE 3. *For the curve* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, *prove that*

$$(i)\ \psi = \frac{\theta}{2} \quad (ii)\ \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}.$$

SOLUTION : Given the equation of the curve

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta).$$

Differentiating this w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta.$$

Now (i) $\quad \frac{dy}{dx} = \frac{dy}{d\theta}\ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 + \cos\theta)}$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\Rightarrow \quad \tan\psi = \tan\frac{\theta}{2} \quad \therefore \ \psi = \frac{\theta}{2}.$$

$$(ii)\ \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}$$

$$= a^2(2 + 2\cos\theta) = 2a^2(1 + \cos\theta)$$

$$= 2a^2.2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$$

$$\therefore \quad \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}.$$

EXAMPLE 4. *Show that in the curve for which* $s = c \log \frac{c}{y}$, *the tangent is of constant length.* **(Bhag. 62S; P.U. 49S, 52S, 61A)**

SOLUTION : From the formula, the length of the tangent

$$= \frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \qquad ...(1)$$

Given the equation of the curve $s = c \log \frac{c}{y}$.

$\Rightarrow$ $s = c(\log c - \log y)$.

$\therefore$ $\frac{ds}{dy} = c\left(0 - \frac{1}{y}\right) = -\frac{c}{y} \Rightarrow \frac{dy}{ds} = -\frac{y}{c} \Rightarrow \sin\psi = -\frac{y}{c}$.

$\therefore$ $\tan\psi = \frac{-y}{\sqrt{c^2 - y^2}}$ i.e. $\frac{dy}{dx} = \frac{-y}{\sqrt{c^2 - y^2}}$.

Therefore from (1), the length of the tangent

$$= \frac{y\sqrt{1 + \frac{y^2}{c^2 - y^2}}}{\frac{y}{\sqrt{c^2 - y^2}}} = \frac{\sqrt{\frac{c^2}{c^2 - y^2}}}{\frac{1}{\sqrt{c^2 - y^2}}}$$

$= c$ (constant)

Hence the result.

Example 16

1. Find $\frac{ds}{dx}$ for the curve

(i) $y^2 = 4ax$ **(R.U. 65S)** (ii) $x^{2/3} + y^{2/3} = a^{2/3}$

(iii) $y = a \cosh \frac{x}{a}$ (iv) $3ay^2 = x(x - a)^2$

2. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $x = a\sin\phi$,

$$\frac{ds}{d\phi} = \sqrt{1 - e^2\sin^2\phi} \qquad \textbf{(B.U. 65S)}$$

3. Show that in the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$$\frac{ds}{dy} = \sqrt{\frac{2a}{y}} \text{ and } \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}. \qquad \textbf{(P.U. 32S)}$$

4. Show that in the curve for which $y^2 = c^2 + s^2$, the perpendicular from the foot of the ordinate upon the tangent is of constant length.

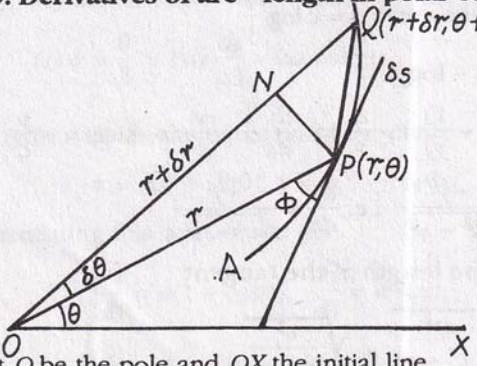
(P.U. 45A, 62A)

ANSWER

1. (i) $\sqrt{\frac{a+x}{x}}$ (ii) $\left(\frac{a}{x}\right)^{1/3}$ (iii) $\frac{y}{a}$ (iv) $\frac{3x+a}{\sqrt{(12ax)}}$

SECTION II. POLAR CO-ORDINATES

5.9. Derivatives of arc—length in polar co-ordinates



Let O be the pole and OX the initial line.

Let P be any point on the curve whose co-ordinates are (r, θ).

Let A be a fixed point on the curve from where the length of the arc AP is s.

Let Q be any other point on the curve which is very near to P. Then the co-ordinates of Q will be $(r + \delta r, \theta + \delta\theta)$,

and arc $AQ = s + \delta s$, $\therefore$ arc $PQ = \delta s$.

From P, PN has been drawn perp. on OQ.

Now, from rt. $\angle ed$ ΔPNQ

$$PN = OP \sin\delta\theta = r\sin\delta\theta; \quad \ldots(1)$$

$$ON = r\cos\delta\theta.$$

$$\therefore \quad NQ = OQ - ON = r + \delta r - r\cos\delta\theta$$

$$= \delta r + r(1 - \cos\delta\theta) = \delta r + r \cdot 2\sin^2\frac{\delta\theta}{2} \quad \ldots(2)$$

Now, if Q, coincides with P i.e. if $\theta \to 0$, then in that case, we can write (1) and (2) as follows :

$$PN = r\left\{\delta\theta - \frac{1}{3!}(\delta\theta)^3 + \ldots\right\}$$

$$= r\delta\theta; \quad \text{up to first order of infinitesimal}$$

$$NQ = \delta r + 2r\left\{\left(\frac{\delta\theta}{2}\right)^2 - \frac{1}{3}\left(\frac{\delta\theta}{2}\right)^4 + \ldots\right\}$$

$$= \delta r.$$

Hence, up to first order of infinitesimal

$$PN = r\delta\theta, \quad NQ = \delta r.$$

Also, $$\frac{\text{chord } PQ}{\text{arc } PQ} \to 1 \text{ when } \delta\theta \to 0.$$

$$\text{chord } PQ \cong \text{arc } PQ = \delta s.$$

Thus in right $\angle ed\ \Delta PNQ$,

$$PN = r\delta\theta, \quad NQ = \delta r, \quad PQ = \delta s.$$

Hence $$PQ^2 = PN^2 + NQ^2$$

$$\Rightarrow \quad \delta s^2 = (r\delta\theta)^2 + (\delta r)^2$$

$$\Rightarrow \quad \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2.$$

Taking the limit,

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

5.10. Angle between Radius vector and tangent

Let the equation of the curve in polar form be $r = f(\theta)$.

Let the co-ordinates of any point P on the curve be (r, θ).

From P, draw a tangent PT which meets OX in T.

Let the angle between the radius vector OP and the tangent PT be ϕ.

[See the fig. of the last article]

Now $$\sin PQN = \frac{PN}{PQ} = \frac{r\sin\delta\theta}{PQ} = r\frac{\sin\delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta s} \cdot \frac{\delta s}{PQ} \qquad \ldots(1)$$

Let $Q \to P$ so that $\angle PQN \to \angle OPT\ (= \phi)$

and $\delta\theta \to 0$ and $$\frac{\delta s}{PQ} = \frac{\text{arc } PQ}{\text{chord } PQ} \to 1$$

Taking the limit of (1),

$$\sin\phi = r \cdot 1 \frac{d\theta}{ds} \cdot 1 = r\frac{d\theta}{ds}$$

Again $\cos PQN = \frac{QN}{PQ} = \frac{\delta r}{PQ} = \frac{\delta r}{\delta s} \cdot \frac{\delta s}{PQ}$ ...(2

$\therefore$ taking the limit,

$$\cos\phi = \frac{dr}{ds}.$$

Again, $\tan PQN = \frac{PN}{QN} = \frac{r\sin\delta\theta}{\delta r} = \frac{\frac{r\sin\delta\theta}{\delta\theta}}{\frac{\delta r}{\delta\theta}}.$

Now taking the limit,

$$\tan\phi = \frac{r}{\frac{dr}{d\theta}} = \frac{rd\theta}{dr}.$$ (P.U. 65

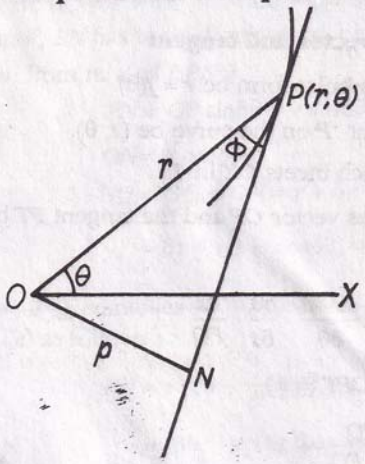
Thus we have proved the following three important formulae which should always be remembered.

(i) $\sin\phi = r\frac{d\theta}{ds}$

(ii) $\cos\phi = \frac{dr}{ds}$

(iii) $\tan\phi = r\frac{d\theta}{dr}$

5.11. Perpendicular from pole on tangent



Let O be the pole and OX the initial line.

Let P be any point on the curve whose co-ordinates are (r, θ). A tangent PN has been drawn to the curve at the point P. From O, draw ON perp. to PN. Then we have to find out the length of ON.

Let $ON = p$. Let $\angle OPN = \phi$.

Then from the rt. $\angle ed\ \Delta ONP$,

$$ON = OP \sin\phi,$$

$$\therefore \quad \boldsymbol{p = r \sin\phi}. \qquad \ldots(1)$$

Now we have to express $\sin\phi$ in terms of θ. For this, we know that

$$\tan\phi = \frac{rd\theta}{dr}$$

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} \Rightarrow \cot^2\phi = \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$\therefore \quad \operatorname{cosec}^2\phi = 1 + \cot^2\phi = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

Now from (1),

$$\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\therefore \quad \boldsymbol{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}. \qquad \ldots(2)$$

Note : If we write $u = \frac{1}{r}$, then $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$\therefore$ from (2), $\boldsymbol{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2}$.

Worked out Examples

EXAMPLE 1. *Show that in the curve $r = ae^{\theta\cot\alpha}$, the angle between the tangent and the radius vector is equal to α.* **(B.U. 58A)**

SOLUTION : Here $r = ae^{\theta\cot\alpha}$.

Taking log on both sides

$$\log r = \log a + \log e^{\theta \cot\alpha} = \log a + \theta \cot\alpha.$$

Again differentiating w.r.t. θ,

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \cot\alpha \Rightarrow \frac{rd\theta}{dr} = \tan\alpha$$

$\Rightarrow \quad \tan\phi = \tan\alpha \quad \therefore \quad \phi = \alpha.$

Hence the result.

Example 17

1. Find $\frac{ds}{d\theta}$ for the following curves

(i) $r = a(1 + \cos\theta)$ (ii) $r = ae^{\theta\cot\alpha}$ (iii) $r = a\theta$

(iv) $r^2 = a^2\cos 2\theta$ (v) $r^m = a^m \cos m\theta$.

2. In the equiangular spiral $r = a\, e^{\theta\cot\alpha}$, prove that $\frac{dr}{ds} = \cos\alpha$. **(B.U. 57S)**

3. Show that $\tan\phi = \dfrac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}$

where ϕ is the angle which the tangent to a curve makes with the radius vector drawn from the origin. **(B.U. 59H)**

[**Hint :** $\psi = \phi + \theta \Rightarrow \phi = \psi - \theta$. $\therefore \quad \tan\phi = \tan(\psi - \theta)$.

Use $\tan\psi = \frac{dy}{dx}$ and $\tan\theta = \frac{y}{x}$.]

4. For the cardiode $r = a(1 - \cos\theta)$, prove that

(i) $\phi = \frac{\theta}{2}$. **(P.U. 30S, 34A, 46A, 47A, 51A, 54A, 58S, 61S; M.U. 65S)**

(ii) $p = 2a \sin^3\frac{\theta}{2}$. **(Bhag. 68S; P.U. 47A; M.U. 67S)**

5. For the parabola $\frac{2a}{r} = 1 - \cos\theta$, prove that

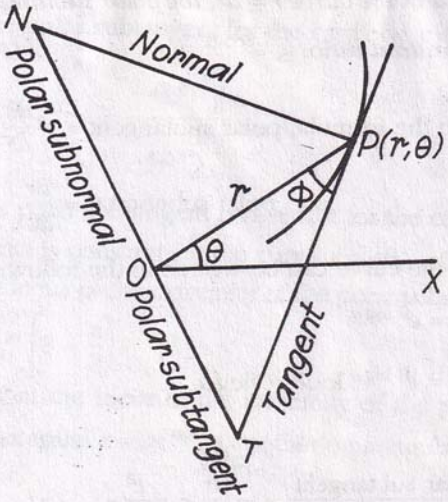
(i) $\phi = \pi - \frac{\theta}{2}$. **(B.U. 63A; M.U. 67S; P.U. 73A)**

(ii) $p = \dfrac{a}{\sin\frac{\theta}{2}}$. **(B.U. 63A)**

Answer

1. (i) $2a\cos\frac{\theta}{2}$ (ii) $a\,\text{cosec}\,\alpha e^{\theta\cot\alpha}$ (iii) $a\sqrt{1+\theta^2}$ (iv) $a\sqrt{\sec 2\theta}$ (v) $\frac{a^m}{r^{m-1}}$.

5.11. Polar Sub-tangent and sub-normal



Let O be the pole and OX the initial line.

Let P be any point on the curve whose co-ordinates are (r, θ).

Let a tangent PT and normal PN be drawn to the curve at the point P.

Again from O, a line NOT has been drawn which is perp to OP. Let us suppose that this line meets the tangent at T and the normal at the point N. Then OT is called the **polar subtangent** and ON the **polar subnormal.**

Let $\angle OPT = \phi$, then

from ΔPOT, $\tan\phi = \dfrac{OT}{OP}$

$$\therefore \quad OT = OP\tan\phi = r\tan\phi$$

$$= r \cdot \frac{rd\theta}{dr} = \frac{r^2 d\theta}{dr}.$$

$$\therefore \textbf{ Polar subtangent} = \frac{r^2 d\theta}{dr}.$$

Again, since $\angle OPT = \phi \quad \therefore \quad \angle OPN = 90° - \phi \quad \therefore \quad \angle ONP = \phi$.

Now, from ΔOPT, $\tan\phi = \dfrac{OP}{ON}$

$$\therefore \quad ON = \frac{OP}{\tan\phi} = \frac{rdr}{rd\theta} = \frac{dr}{d\theta}.$$

$$\therefore \textbf{ Polar subnormal} = \frac{dr}{d\theta}.$$ **(B.U. 59A, 63S)**

EXAMPLE 1. *Show that in the curve $r = a^\theta$, the polar subtangent and the polar subnormal have a constant ratio.* **(P.U. 44S; B.U. 56A)**

SOLUTION : From the formula, polar subtangent $= r^2\frac{d\theta}{dr}$

and polar subnormal $= \frac{dr}{d\theta}$.

The equation of the curve can be written in the following way:

$$r = e^{\theta \log a}$$

$$\therefore \quad \frac{dr}{d\theta} = e^{\theta \log a}.\log a = r\log a.$$

$$\therefore \quad \frac{\text{polar subtangent}}{\text{polar subnormal}} = \frac{r^2\frac{d\theta}{dr}}{\frac{dr}{d\theta}} = \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{r^2}{r^2(\log a)^2} = \frac{1}{(\log a)^2} = \text{const.}$$

Hence the result.

Note : Writing e in place of a, the equation of the curve becomes $r = e^\theta$.

In that case $\frac{\text{polar subtangent}}{\text{polar subnormal}} = \frac{1}{(\log e)^2} = 1.$

Hence polar subtangent = polar subnormal.

Example 18

1. Show that for the curve $r = e^\theta$, the polar subtangent is equal to the polar subnormal. **(R.U. 68A)**

2. Prove that in the parabola $\frac{2a}{r} = 1 - \cos\theta$, the polar subtangent $= 2a\,\text{cosec}\,\theta$.

3. In the curve $r(1 + \cos\theta) = 2a$, prove that the polar subtangent $= 2a\,\text{cosec}\,\theta$.

4. Prove that the polar subtangent for the cardiode $r = a(1 - \cos\theta)$ is

$$= \frac{2a\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}}.$$ **(P.U. 33A, 53A)**

5. Show that the polar subtangent is constant for the curve $r\theta = a$ and the polar subnormal is constant for the curve $r = a\theta$. **(P.U. 38S; M.U. 68A)**

6. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'\left(\theta - \frac{\pi}{2}\right)$.

Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral.

(Bhag. 91H, 94H)

[**Soln.** : See fig. of Art. 5.11.

Let $P(r, \theta)$ be the point on the curve and let N be the end-point of the polar subnormal.

Let the co-ordinates of N be (r_1, θ_1).

We have to find the locus of N.

$r_1 = ON =$ polar subnormal

$$= \frac{dr}{d\theta} = f'(\theta). \qquad \ldots(1)$$

Now we want to express θ in terms of θ_1.

For this $\theta_1 = \angle XON = \frac{\pi}{2} + \theta \quad \therefore \quad \theta = \theta_1 - \frac{\pi}{2}$.

Substituting this value of θ in (1), we get

$$r_1 = f'\left(\theta_1 - \frac{\pi}{2}\right).$$

Hence the locus of $N(r_1, \theta_1)$ is

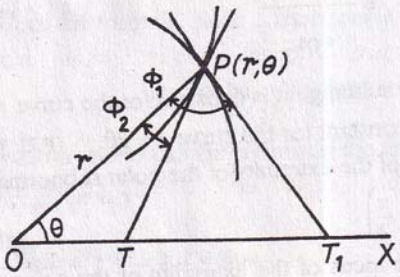
$$r = f'\left(\theta - \frac{\pi}{2}\right).$$

The second part follows as a consequence.]

7. Prove that the locus of the extremity of the polar subtangent of the curve $\frac{1}{r} = f(\theta)$ is $\frac{1}{r} + f'\left(\theta + \frac{\pi}{2}\right) = 0$.

8. Prove that the locus of the extremity of the polar subtangent of the curve $\frac{1}{r} + f(\theta) = 0$ is $\frac{1}{r} = f'\left(\frac{\pi}{2} + \theta\right)$.

5.13. Angle of intersection of two curves whose polar equations are given.



Let the equations of the two curves be $r = f(\theta)$ and $r = F(\theta)$.

Let P be the point of intersection of both the curves.

We know that the angle of intersection of two curves is the same as the angle between their tangents at their point of intersection.

Let the tangents PT and PT_1 be drawn at the point of intersection of the two curves.

Let $\angle OPT = \phi_1$ and $\angle OPT_1 = \phi_2$.

$$\therefore \quad \tan\phi_1 = \frac{r d\theta}{dr} = r\frac{1}{\frac{dr}{d\theta}} = \frac{f(\theta)}{f'(\theta)}$$

and $$\tan\phi_2 = \frac{F(\theta)}{F'(\theta)}.$$

Let the angle between the two curves i.e. the angle between their tangents be $= \alpha$.

Then $$\alpha = \phi_1 - \phi_2.$$

$$\therefore \quad \tan\alpha = \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 . \tan\phi_2}$$

$$= \frac{\frac{f(\theta)}{f'(\theta)} - \frac{F(\theta)}{F'(\theta)}}{1 + \frac{f(\theta)}{f'(\theta)} \cdot \frac{F(\theta)}{F'(\theta)}}$$

$$= \frac{f(\theta)F'(\theta) - f'(\theta)F(\theta)}{f'(\theta)F'(\theta) + f(\theta)F(\theta)}.$$

Cor. (i) If the two curves touch each other, then $\alpha = 0$.

$$\therefore \quad f(\theta)F'(\theta) - f'(\theta)F(\theta) = 0$$

$$\therefore \quad \frac{f(\theta)}{f'(\theta)} = \frac{F(\theta)}{F'(\theta)}.$$

(ii) If the two curves cut each other at rt. angle, then $\alpha = \frac{\pi}{2}$.

$$\therefore \quad f'(\theta)F'(\theta) + f(\theta)F(\theta) = 0.$$

Note : In practical problems, we find out the angle between two curves by first evaluating $\tan\phi_1$, $\tan\phi_2$ and then using $\alpha = \phi_1 - \phi_2$.

Worked Out Examples

EXAMPLE 1. *Find the angle of intersection of the curves*

$$r = a(1 + \cos\theta), \quad r = b(1 - \cos\theta).$$

(P.U. 59A, 60A, 62A, 68A; B.U. 64A)

SOLUTION : Given $r = a(1 + \cos\theta)$ ...(1)

$$r = b(1 - \cos\theta). \qquad \ldots(2)$$

Taking logarithm of (1),

$$\log r = \log a + \log(1 + \cos\theta).$$

Then differentiating w.r.t. θ, we have

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1 + \cos\theta}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

$$\Rightarrow \qquad r\frac{d\theta}{dr} = -\cot\frac{\theta}{2} \quad \therefore \quad \tan\phi_1 = -\cot\frac{\theta}{2}.$$

Again taking logarithm of (2),

$$\log r = \log b + \log(1 - \cos\theta).$$

Then differentiating w.r.t. θ, we have

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1 - \cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$\Rightarrow \quad \frac{rd\theta}{dr} = \tan\frac{\theta}{2}. \qquad \therefore \quad \tan\phi_2 = \tan\frac{\theta}{2}.$$

Here $\tan\phi_1.\tan\phi_2 = -\cot\frac{\theta}{2}\tan\frac{\theta}{2} = -1.$

Hence the two curves cut each other at right angle.

EXAMPLE 2. *Find the angle of intersection of the curves*

$$r^2 = 16 \sin 2\theta \quad \textit{and} \quad r^2\sin2\theta = 4.$$

SOLUTION : First we take the curve $r^2 = 16 \sin 2\theta$.

Taking logarithm of both sides, we have

$$2 \log r = \log 16 + \log \sin 2\theta.$$

Then differentiating w.r.t. θ, we have

$$2\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{2\cos2\theta}{\sin2\theta} = 2\cot2\theta$$

$$\Rightarrow \quad \tan\phi_1 = \tan2\theta \quad \therefore \quad \phi_1 = 2\theta.$$

Now we take the curve $r^2 \sin2\theta = 4$.

Taking log of both sides,

$$2 \log r + \log \sin2\theta = \log4.$$

Then differentiating w.r.t. θ, we have

$$2\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{2\cos2\theta}{\sin2\theta} = 0$$

$$\Rightarrow \quad \frac{1}{r}\cdot\frac{dr}{d\theta} = -\cot2\theta$$

$$\Rightarrow \quad \tan\phi_2 = -\tan2\theta = \tan(\pi - 2\theta)$$

$$\therefore \quad \phi_2 = \pi - 2\theta.$$

Hence the angle between the two curves

$$\phi_1 \sim \phi_2 = (\pi - 2\theta) - 2\theta = \pi - 4\theta.$$

Now we find the value of θ at the point of intersection of the two curves.

For this, from the equation of the curve $r^2 = 16 \sin2\theta = \frac{4}{\sin2\theta}$

$$\Rightarrow \quad \sin^2 2\theta = \frac{4}{16} = \frac{1}{4}$$

$$\Rightarrow \quad \sin2\theta = \frac{1}{2} \quad \therefore \quad 2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}.$$

$\therefore$ The reqd. angle between the two curves $= \pi - 4\theta$

$$= \pi - \frac{4\pi}{12} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Example 19

1. Show that the following curves cut orthogonally

(i) $r = \dfrac{a}{1+\cos\theta}$, $r = \dfrac{b}{1-\cos\theta}$.

(ii) $r^m = a^m \cos m\theta$, $r^m = b^m \sin m\theta$. **(R.U. 71S)**

2. Show that the curves $r^n = a^n \sec(n\theta + \alpha)$, $r^n = b^n \sec(n\theta + \beta)$ intersect at an angle which is independent of a and b. ***(B.U. 72A)***

3. Find the angle between the curves

(i) $r = a\theta$, $r\theta = a$ (ii) $r = ae^{\theta}$, $re^{\theta} = b$

(iii) $r = a(1 - \cos\theta)$ and $r = 2a\cos\theta$ **(M.U. 72A)**

(iv) $r^2 = a^2 \cos 2\theta$, $r^2 = a^2 \sin 2\theta$.

Answer

3. (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{1}{2}\left[\pi + \cos^{-1}\frac{1}{3}\right]$ (iv) $\frac{\pi}{2}$

SECTION III. PEDAL EQUATION

5.14. The (p, r) or Pedal Equation of curve

So far, we have been acquainted with two types of equation of any curve : one cartesian equation containing (x, y) and the other the polar equation containing (r, θ). When the equation of any curve is given in terms of (p, r) where p is the length of the perpendicular from the pole on the tangent and r is the radius vector, then that form of the curve is called the pedal equation. Now, we shall illustrate one-by-one, when the equation of the curve is given either in cartesian form or polar form, how to transform that equation in pedal form i.e. in the form of (p, r).

(i) To find the pedal equation of a curve from its polar form.

Let the polar equation of any curve be

$$f(r, \theta) = 0 \qquad \ldots(1)$$

Let the co-ordinates of any point on the curve be (r, θ) and let the length of the perpendicular from the pole on the tangent at (r, θ) be p.

If ϕ be the angle between the tangent and radius vector, then we know that

$$\tan\phi = r\frac{d\theta}{dr} \qquad \ldots(2)$$

and $p = r\sin\phi$ (3)

Now, if we eliminate θ between the equations (1), (2) and (3), then we shall get an equation in terms of p and r and this will be the required pedal equation of the curve.

Note : Sometines we easily get the pedal equation of the curve by eliminating θ between the equations

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

and $r = f(\theta)$.

(ii) To find the pedal equation of a curve from its cartesian equation.

Let the cartesian equation of any curve be

$$f(x, y) = 0. \qquad ...(1)$$

Then the equation of the tangent at the point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x).$$

If the length of the perpendicular on the tangent from the origin (0, 0) be p, then

$$p = \frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \qquad ...(2)$$

Also $r^2 = x^2 + y^2$. ...(3)

Now, if we eliminate x and y from the equations (1), (2) and (3), then we shall get an equation in terms of p and r and this will be the required pedal equation of the curve.

Worked Out Examples

EXAMPLE 1. *Show that the pedal equation of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *w.r.t. its centre is*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

(B.U. 55S, 70A; M.U. 64S, 68A, 81; P.U. 69S)

SOLUTION : Given : The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

The equation of the tangent at the point (x, y) is $\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1.$

Now, the length of the perpendicular on the tangent from the centre (0, 0) is

$$p = \frac{1}{\sqrt{\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2}}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}. \qquad \ldots(2)$$

Also $r^2 = x^2 + y^2$. $\qquad \ldots(3)$

Now, we shall have to eliminate x and y from (1), (2) and (3) for finding out the pedal equation of the given curve.

Eliminating x^2 and y^2 from (1), (2) and (3), we get

$$\begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & 1 \\ \frac{1}{a^4} & \frac{1}{b^4} & \frac{1}{p^2} \\ 1 & 1 & r^2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a^2 & b^2 & p^2 \\ 1 & 1 & 1 \\ a^4 & b^4 & r^2p^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} a^2 & b^2 & p^2 \\ a^4 & b^4 & r^2p^2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Expanding the above determinant, we get

$$p^2(a^4 - b^4) - r^2p^2(a^2 - b^2) + 1(a^2b^4 - a^4b^2) = 0$$

$$\Rightarrow \quad p^2(a^2 + b^2) - r^2p^2 - a^2b^2 = 0$$

$$\Rightarrow \quad a^2b^2 = p^2(a^2 + b^2 - r^2)$$

$$\Rightarrow \quad \frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

which is the reqd. pedal equation.

EXAMPLE 2. *Find the pedal equation of the parabola $y^2 = 4ax$ w.r.t. its focus.*

(P.U. 57; M.U. 64A, 70A, 86)

SOLUTION : Given that the equation of the parabola is $y^2 = 4ax$.

We know that the focus of the parabola is $(a, 0)$. By transferring the origin to the focus $(a, 0)$, the equation of the parabola becomes

$$(y + 0)^2 = 4a(x + a) \Rightarrow y^2 = 4a(x + a),$$

$$\Rightarrow y^2 = 4ax + 4a^2.$$

We know, from Co-ordinate Geometry that the equation of the tangent to the parabola (1) at the point (x, y) is

$$Yy = 2a(X + x) + 4a^2 \Rightarrow 2aX - Yy + 4a^2 + 2ax = 0.$$

Now, p = the length of the perpendicualr drawn from the origin (0, 0) upon this tangent

$$= \frac{4a^2 + 2ax}{\sqrt{4a^2 + y^2}}$$

$$\Rightarrow \quad p = \frac{4a^2 + 2ax}{\sqrt{\{4a^2 + 4ax + 4a^2\}}} \text{ ; from (1)}$$

$$\Rightarrow \quad p = \frac{4a^2 + 2ax}{\sqrt{\{8a^2 + 4ax\}}} = \frac{2a(2a + x)}{\sqrt{4a(2a + x)}}$$

$$= \frac{2a(2a + x)}{2\sqrt{a}\,\sqrt{2a + x}} = \sqrt{a} \cdot \sqrt{(2a + x)}$$

$$\Rightarrow \quad p^2 = a(2a + x). \qquad ..(2)$$

Also, $r^2 = x^2 + y^2 = x^2 + 4ax + 4a^2 = (x + 2a)^2$

i.e. $r = 2a + x.$ ...(3)

Hence from (2) and (3), we get $p^2 = ar$

which is the required pedal equation.

EXAMPLE 3. *Find the pedal equation of the curve* $x = a\cos^3\theta$, $y = a\sin^3\theta$.

SOLUTION : Here $x = a\cos^3\theta$ ...(1)

and $y = a\sin^3\theta.$ ...(2)

Differentiating (1) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(3\cos^2\theta)(-\sin\theta) = -3a\sin\theta\cos^2\theta.$$

Similarly differentiating (2) w.r.t. θ, we get

$$\frac{dy}{d\theta} = a.(3\sin^2\theta)\cos\theta = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\sin\theta\cos^2\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

Now, the equation of the tangent is,

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(X - a\cos^3\theta)$$

$$\Rightarrow \quad Y\cos\theta - a\sin^3\theta\cos\theta = -X\sin\theta + a\sin\theta\cos^3\theta$$

$$\Rightarrow \quad X\sin\theta + Y\cos\theta = a\sin\theta\cos^3\theta + a\sin^3\theta\cos\theta$$

$$= a\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta) = a\sin\theta\cos\theta.$$

$\therefore$ p = the length of the perpendicular from the origin (0, 0) on this tangent

$$= \frac{a\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = a\sin\theta\cos\theta. \qquad ...(3)$$

Also, $r^2 = x^2 + y^2$

$$= a^2\cos^6\theta + a^2\sin^6\theta = a^2(\cos^6\theta + \sin^6\theta)$$

$$= a^2\{(\cos^2\theta)^3 + (\sin^2\theta)^3\}$$

$$= a^2\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\}$$

$$= a^2\{1 - 3\cos^2\theta\sin^2\theta\} = a^2\left\{1 - 3\cdot\frac{p^2}{a^2}\right\};\ \text{from (3)}$$

$$= a^2 - 3p^2$$

which is the required pedal equation.

EXAMPLE 4. *Show that the pedal equation of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *with regard to its focus is* $\frac{b^2}{p^2} = \frac{2a}{r} - 1$. **(Bhag. 93H)**

SOLUTION : We know that the focus of the ellipse is $(ae, 0)$. By transferring the origin to the focus $(ae, 0)$, the equation of the ellipse becomes.

$$\frac{(x + ae)^2}{a^2} + \frac{(y + 0)^2}{b^2} = 1 \Rightarrow \frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

The co-ordinates of any point on the ellipse (1) is

$$x + ae = a\cos\theta \text{ and } y = b\sin\theta$$

i.e. $$x = a\cos\theta - ae \text{ and } y = b\sin\theta.$$

Now $$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta.$$

Therefore the equation of the tangent at (x, y) is

$$y - b\sin\theta = -\frac{b}{a}\cot\theta\{x - (a\cos\theta - ae)\}$$

$$\Rightarrow \quad ay - ab\sin\theta = -b\cot\theta\cdot x + b\cot\theta(a\cos\theta - ae)$$

$$\Rightarrow \quad b\cot\theta\cdot x + ay = ab\sin\theta + b\cot\theta(a\cos\theta - ae).$$

Hence p = length of the perpendicular from the origin on the tangent

$$= \frac{ab\sin\theta + b\cot\theta(a\cos\theta - ae)}{\sqrt{a^2 + b^2\cot^2\theta}}$$

$$= \frac{[ab\sin\theta + b\cot\theta(a\cos\theta - ae)]\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= \frac{ab\sin^2\theta + b\cos\theta(a\cos\theta - ae)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= \frac{ab - abe\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab - abe\cos\theta)^2} \qquad ...(2)$$

Now $r^2 = x^2 + y^2$

$$= (a\cos\theta - ae)^2 + b^2\sin^2\theta$$

$$= a^2(\cos^2\theta - 2e\cos\theta + e^2) + b^2\sin^2\theta$$

$$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2(1 - \cos^2\theta)$$

$$= (a^2 - b^2)\cos^2\theta - 2a^2e\cos\theta + (a^2 - b^2) + b^2$$

$$= a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$$

$$= a^2(1 - e\cos\theta)^2$$

$$\therefore \quad r = a(1 - e\cos\theta) \qquad ...(3)$$

The num. of (2) is $a^2\sin^2\theta + b^2\cos^2\theta$

$$= a^2(1 - \cos^2\theta) + b^2\cos^2\theta$$

$$= a^2 - \cos^2\theta(a^2 - b^2)$$

$$= a^2 - a^2e^2\cos^2\theta = a^2(1 - e^2\cos^2\theta)$$

$$= a^2(1 + e\cos\theta)(1 - e\cos\theta)$$

$$= a^2\left\{1 + \frac{a - r}{a}\right\}\frac{r}{a} = r(2a - r)\ ;\ \text{from (3).}$$

The denom. of (2) is $= (ab - abe\cos\theta)^2$

$$= a^2b^2(1 - e\cos\theta)^2$$

$$= a^2b^2\left(1 - \frac{a - r}{a}\right)^2 = a^2b^2\frac{r^2}{a^2} = b^2r^2.$$

$\therefore$ From (2), $\dfrac{1}{p^2} = \dfrac{r(2a - r)}{b^2 r^2} = \dfrac{2a - r}{b^2 r}$

$\Rightarrow \qquad \dfrac{b^2}{p^2} = \dfrac{2a - r}{r} = \dfrac{2a}{r} - 1.$

Hence the result.

EXAMPLE 5. *Find the pedal equation of* $r^m = a^m \cos m\theta$.

(Bhag. 66A, 92; R.U. 66A, 68S, 72S; B.U. 59S, 61S, 64S, 76A; M.U. 66A, 74A, 80, 90H; P.U. 50A)

SOLUTION : Taking log of both sides of the given equation

$$m \log r = m \log a + \log \cos m\theta.$$

Now, differentiating both sides w.r.t. θ, we get

$$m \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}(-\sin m\theta) \times m = - m \tan m\theta.$$

$$\therefore \qquad \frac{1}{r}\frac{dr}{d\theta} = -\tan m\theta \Rightarrow \frac{r d\theta}{dr} = - \cot m\theta$$

$$\Rightarrow \qquad \tan\phi = - \cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore \qquad \phi = \frac{\pi}{2} + m\theta. \qquad \ldots(1)$$

Again, $p = r\sin\phi = r \sin\left(\dfrac{\pi}{2} + m\theta\right)$; from (1),

$$= r \cos m\theta.$$

But from the given equation, $\cos m\theta = \dfrac{r^m}{a^m}$.

$$\therefore \qquad p = r \cdot \frac{r^m}{a^m} = \frac{r^{m+1}}{a^m}.$$

Cor. Putting $m = 2$, the pedal equation of $r^2 = a^2 \cos 2\theta$ will be

$$p = \frac{r^3}{a^2} \quad \therefore \quad r^3 = a^2 p.$$

(M.U. 65A; Bhag. 65A; R.U. 65A; B.U. 53A; P.U. 32A; 38A, 55S, 61A, 65S, 69A)

EXAMPLE 6. *Find the pedal equation of the cardiode* $r = a(1 - \cos\theta)$.

(P.U. 30S, 34, 46A, 51A, 54A, 58S, 64S, 68S, 86; B.U. 68A; Bhag. 63A; M.U. 65S)

SOLUTION : Taking log of both sides of the given equation,

$$\log r = \log a + \log(1 - \cos\theta).$$

Now differentiating both sides w.r.t. θ,

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{1 - \cos\theta} \times \sin\theta$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = \frac{1 - \cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\Rightarrow \quad \tan\phi = \tan\frac{\theta}{2} \quad \therefore \ \phi = \frac{\theta}{2} \qquad \ldots(1)$$

Again $p = r\sin\phi = r\sin\frac{\theta}{2}$; from (1).

Squaring,

$$p^2 = r^2 \sin^2\frac{\theta}{2} = r^2 \cdot \frac{1 - \cos\theta}{2}$$

$$= \frac{r^2}{2} \cdot \frac{r}{a}; \text{ from the eqn. of the curve}$$

$$= \frac{r^3}{2a}.$$

This is the reqd. pedal equation.

EXAMPLE 7. *In the curve* $\frac{2a}{r} = 1 - \cos\theta \{i.e. r(1 - \cos\theta) = 2a\}$, *prove that* $p^2 = ar$.

(Bhag. 62S, 72A; B.U. 54S, 63A, 71A; P.U. 47S, 48A, 49S, 52S, 54S, 59S, 62S, 68A, 70S, 73A, 75A; R.U. 70A, 73A, 81; M.U.79, 87)

SOLUTION : Taking log of both sides of the given equation,

$$\log 2a - \log r = \log(1 - \cos\theta).$$

Now differentiating both sides w.r. to θ,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1 - \cos\theta} \times \sin\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1 - \cos\theta}$$

$$\therefore \quad r\frac{d\theta}{dr} = -\frac{1 - \cos\theta}{\sin\theta} = -\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

$$\Rightarrow \qquad \tan\phi = -\tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$$

$$\therefore \qquad \phi = \pi - \frac{\theta}{2}. \qquad \ldots(1)$$

Again, $p = r\sin\phi$

$$= r\sin\left(\pi - \frac{\theta}{2}\right) = r\sin\frac{\theta}{2}; \text{ from (1).}$$

Squaring,

$$p^2 = r^2\sin^2\frac{\theta}{2} = r^2.\frac{1-\cos\theta}{2}$$

$$= \frac{r^2}{2}.\frac{2a}{r}; \text{ since from the given equation } 1 - \cos\theta = \frac{2a}{r}$$

$$= ar.$$

Hence the result.

Example 20

Show that the pedal equation of the curve

1. $r^m = a^m \sin m\theta$ is $r^{m+1} = a^m p$. **(P.U. 41S, 44A ; Bhag. 94)**

2. (i) $r^2 = a^2\cos 2\theta$ is $a^2p = r^3$. **(P.U. 32A, 38A, 46S, 55S, 63S, 65A, 69A; B.U. 53A; Bhag. 65A; R.U. 65A; M.U.71S)**

   (ii) $r^2 = a^2\sin 2\theta$ is $a^2p = r^3$. **(P.U.46)**

   (iii) $r^5 = 32\cos 5\theta$ is $32p = r^6$. **(M.U.69)**

3. $r = ae^{m\theta}$ is $p\sqrt{1+m^2} = r$. **(P.U.52A; B.U. 56S)**

4. $r = ae^{\theta\cot\alpha}$ is $p = r\sin\alpha$. **(P.U. 51S, 57S; B.U. 57S)**

5. $r = a\theta$ is $p^2 = \dfrac{r^4}{a^2 + r^2}$. **(P.U. 53S; Bhag. 63A; M.U.73 A, 75A)**

6. $r\theta = a$ is $p^2(a^2 + r^2) = a^2r^2$.

7. $r\sin n\theta = a$ is $p^2 = \dfrac{a^2r^2}{a^2 + n^2(r^2 - a^2)}$. **(P.U. 39A, 53S)**

8. $r\cos\dfrac{\sqrt{a^2-b^2}}{a}\theta = \sqrt{a^2-b^2}$ is $p\sqrt{b^2+r^2} = a\,r$. **(P.U. 50S)**

9. Show that the pedal equation of the spiral $r = a\sec h n\theta$ is of the form $\dfrac{1}{p^2} = \dfrac{A}{r^2} + B$.

10. Find the pedal equation of the conic $\dfrac{l}{r} = 1 + e\cos\theta$. **(M.U.74H)**

11. Show that the pedal equation of the parabola

$y^2 = 4a(x + a)$ is $p^2 = ar$. **(P.U.76A; M.U. 67A; R.U. 63A, 67S, 75A, 82)**

12. Show that the pedal equation of the astroid

$x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 + 3p^2 = a^2$. **(M.U. 72S; B.U.72H; R.U. 74A)**

13. Find the pedal equation of the curve $x^2 - y^2 = a^2$. **(P.U. 71A)**

14. Show that the pedal equation of the curve

$x = a e^{\theta} (\sin\theta - \cos\theta)$, $y = a e^{\theta}(\sin\theta + \cos\theta)$ is $r = \sqrt{2}p$.

(M.U.71A; Bhag. 90H)

15. Find the pedal equation of the parabola $y^2 = 4ax$ .w.r.t. its vertex.

(P.U.74H)

16. Show that the pedal equation of the ellipse $a^2x^2 + b^2y^2 = 1$ w.r.t. the centre is

$$\begin{vmatrix} 1 & 1 & r^2 \\ a^2 & b^2 & 1 \\ a^4 & b^4 & p^{-2} \end{vmatrix} = 0.$$ **(B.U. 62A)**

17. Show that for the epicycloid given by

$$x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta$$

$$p = (a + 2b)\sin\frac{a}{2b}\theta \text{ & } \psi = \frac{a+2b}{2b}\theta.$$

Also, obtain the pedal equation of this curve. **(Bhag. 92 H)**

[**Hints :** $\frac{dy}{d\theta} = (a + b)\cos\theta - b.\cos\frac{a+b}{b}\theta . \frac{a+b}{b}$

$= (a + b)\cos\theta - (a + b).\cos\frac{a+b}{b}\theta$

$\frac{dx}{d\theta} = -(a + b)\sin\theta + (a + b)\sin\frac{a+b}{b}\theta.$

$\therefore \quad \tan\psi = \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

$= \frac{\cos\theta - \cos\frac{a+b}{b}\theta}{\sin\frac{a+b}{b}\theta - \sin\theta} = \frac{2\sin\frac{a+2b}{2b}\theta \cdot \sin\frac{a}{2b}\theta}{2\cos\frac{a+2b}{2b}\theta \sin\frac{a}{2b}\theta} = \tan\frac{a+2b}{2b}\theta$

$\Rightarrow \quad \psi = \frac{a+2b}{2b}\theta$

The equation of tangent at (x,y) is

$$Y - y = \frac{dy}{dx}(X - x).$$

$\therefore$ $p \equiv$ length of the perpendicular from the origin to the tangent

$$= \frac{(x \tan \alpha - y)}{\sqrt{1 + \tan^2 \alpha}}, \quad \text{where} \quad \alpha = \frac{a + 2b}{2b}\theta$$

$$= x \sin\alpha - y \cos\alpha \quad \text{etc.}]$$

ANSWERS

10. $\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)$ **13.** $pr = a^2$

15. $(r^2 - p^2)^2 a^2 = p^2(p^2 + 4a^2)(r^2 + 4a^2)$.

5.15. Polar Equation of tangent

Let $u = \frac{1}{r} = A\cos\theta + B\sin\theta$ ...(1)

be a chord passing through two points P and Q corresponding to $\theta = \alpha$ and $\alpha + \delta\alpha$ on the curve $u = f(\theta)$. ...(2)

That the equation (1) represents a st. line where A and B are arbitrary constants, can be seen by converting it to cartesian co-ordinates by the substitution $x = r\cos\theta$ and $y = r\sin\theta$. At the points P and Q, the values of r corresponding to (1) and (2) are the same.

$\therefore$ At $\theta = \alpha$, from (1) and (2),

$$f(\alpha) = A\cos\alpha + B\sin\alpha \qquad ...(3)$$

and $$f(\alpha + \delta\alpha) = A\cos(\alpha + \delta\alpha) + B\sin(\alpha + \delta\alpha) \qquad ...(4)$$

On subtraction,

$$f(\alpha + \delta\alpha) - f(\alpha) = A\{\cos(\alpha + \delta\alpha) - \cos\alpha\} + B\{\sin(\alpha + \delta\alpha) - \sin\alpha\}$$

$$\Rightarrow \frac{f(\alpha + \delta\alpha) - f(\alpha)}{\delta\alpha} = A\frac{\cos(\alpha + \delta\alpha) - \cos\alpha}{\delta\alpha} + B\frac{\sin(\alpha + \delta\alpha) - \sin\alpha}{\delta\alpha}$$

Hence in the limit when $\delta\alpha \to 0$,

$$f'(\alpha) = A(\text{d.c. of } \cos\alpha) + B(\text{d.c. of } \sin\alpha)$$

i.e. $$\frac{du}{d\alpha} = -A\sin\alpha + B\cos\alpha.$$

Also, in the limit when $\delta\alpha \to 0$, the chord (1) becomes the tangent to the curve at the point $\theta = \alpha$.

Thus if $u = A\cos\theta + B\sin\theta$ is a tangent to the curve $u = f(\theta)$, then the value of $\frac{du}{d\theta}$ at $(\theta = \alpha)$ will the same for the curve and the tangent.

Putting $\theta = \alpha$ in (1) and (2), we have

$$U = A\cos\alpha + B\sin\alpha \qquad ...(5)$$

and $$U' = -A\sin\alpha + B\cos\alpha. \qquad ...(6)$$

Here U is the value of u when $\theta = \alpha$.

Solving (5) and (6), we get

$$A = U\cos\alpha - U'\sin\alpha$$

$$B = U\sin\alpha + U'\cos\alpha.$$

By substituting the values of A and B, in (1), the equation of the tangent is $\boldsymbol{u = U\cos(\theta - \alpha) + U'\sin(\theta - \alpha).}$

Note : We could have also taken the equation of any st. line in polar, form as $u = p\cos(\theta - \alpha) + q\sin(\theta - \alpha)$ where p and q are arbitrary constants.

5. 16. Polar equation of the normal

We have seen that the polar equation of the tangent at $\left(\frac{1}{U}, \theta_1\right)$ is

$$u = U\cos(\theta - \alpha) + U'\sin(\theta - \alpha) \qquad ...(1)$$

Hence the equation of any line perpendicular to (1) is given by

$$Au = U\cos\left(\theta + \frac{\pi}{2} - \alpha\right) + U'\sin\left(\theta + \frac{\pi}{2} - \alpha\right)$$

$$= -U\sin(\theta - \alpha) + U'\cos(\theta - \alpha) \qquad ...(2)$$

where A is any arbitrary constant.

If (2) is the required normal, then it must pass through (U, α).

Therefore, from (2),

$$AU = U' \times 1 - U(0) \Rightarrow A = U'/U.$$

Hence putting the value of A in (2), we will have the required normal

$$\frac{U'}{U}u = \boldsymbol{U'\cos(\theta - \alpha) - U\sin(\theta - \alpha).}$$

EXAMPLE 1. *Find the equation of the tangent and normal at the point* $\theta = \alpha$ *on the cardiode* $r = a(1 + \cos\theta)$.

SOLUTION : Here $u = \frac{1}{r} = \frac{1}{a(1 + \cos\theta)} = \frac{1}{a \cdot 2\cos^2\frac{\theta}{2}} = \frac{1}{2a}\sec^2\frac{\theta}{2}.$

$$\therefore \quad \frac{du}{d\theta} = \frac{1}{2a} \cdot 2\sec^2\frac{\theta}{2}\tan\frac{\theta}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{2a} \cdot \sec^2\frac{\theta}{2} \cdot \tan\frac{\theta}{2}$$

Therefore at $\theta = \alpha$,

$$U = \frac{1}{2a}\sec^2\frac{\alpha}{2} \text{ and } U' = \frac{1}{2a}\sec^2\frac{\alpha}{2}\tan\frac{\alpha}{2}.$$

Hence (i) the equation of the tangent at $\theta = \alpha$ is

$$u = U\cos(\theta - \alpha) + U'\sin(\theta - \alpha)$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\cos(\theta - \alpha) + \frac{1}{2a}\sec^2\frac{\alpha}{2}\tan\frac{\alpha}{2}\sin(\theta - \alpha)$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\left[\cos(\theta - \alpha) + \tan\frac{\alpha}{2}\sin(\theta - \alpha)\right]$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\left[\frac{\cos(\theta - \alpha)\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta - \alpha)}{\cos\frac{\alpha}{2}}\right]$$

$$= \frac{\sec^2\frac{\alpha}{2}}{2a\cos\frac{\alpha}{2}}\left[\cos\left(\theta - \alpha - \frac{\alpha}{2}\right)\right]$$

$$= \frac{1}{2a} \cdot \frac{1}{\cos^3\alpha/2}\cos\left(\theta - \frac{3\alpha}{2}\right).$$

This becomes $x\cos\frac{3\alpha}{2} + y\sin\frac{3\alpha}{2} = 2a\cos^3\frac{\alpha}{2}$ in cartesian co-ordinates.

(ii) The equation of the normal of $\theta = \alpha$ is

$$u = \frac{U}{U'}[U'\cos(\theta - \alpha) - U\sin(\theta - \alpha)]$$

$$= U\cos(\theta - \alpha) - \frac{U^2}{U'}\sin(\theta - \alpha)$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\cos(\theta - \alpha) - \frac{1}{4a^2} \times \frac{\sec^4\frac{\alpha}{2} \times 2a}{\sec^2\frac{\alpha}{2}\tan\frac{\alpha}{2}}\sin(\theta - \alpha)$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\cos(\theta - \alpha) - \frac{1}{2a}\frac{\sec^2\frac{\alpha}{2}}{\tan\frac{\alpha}{2}}\sin(\theta - \alpha)$$

$$= \frac{1}{2a}\sec^2\frac{\alpha}{2}\left[\cos(\theta - \alpha) - \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin(\theta - \alpha)\right]$$

$$= \frac{1}{2a}\cdot\frac{\sec^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\left[\cos(\theta - \alpha)\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\sin(\theta - \alpha)\right]$$

$$= \frac{1}{2a}\frac{\sec^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\left\{\frac{\alpha}{2} - (\theta - \alpha)\right\}$$

$$= \frac{1}{2a}\frac{\sec^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\left(\frac{3\alpha}{2} - \theta\right).$$

This becomes $x\sin\frac{3\alpha}{2} - y\cos\frac{3\alpha}{2} = a\sin\alpha\cos\frac{\alpha}{2}$ in cartesian co-ordinates.

EXAMPLE 2. *Prove that the locus of the intersection of the normal at the ends of a focal chord of the cardiode $r = a(1 + \cos\theta)$ is a circle.*

(P.U. 56H, 64H; B.U. 59H, 64H; Bhag. 63H, 65H, 69H, 89H, R.U. 62H; M.U. 64H, 72H, 81H)

SOLUTION : Let CD be a focal chord of the curve.

Let the vectorial angle of C be α. Then the vectorial angle of D will be $\pi + \alpha$.

Now, the equation of normal at C i.e. $\theta = \alpha$, from Ex. 1 is

$$u = \frac{1}{2a}\cdot\frac{\sec^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\left(\frac{3\alpha}{2} - \theta\right) \quad ...(1)$$

i.e. $$\frac{1}{r} = \frac{1}{2a} \cdot \frac{\sec^2\frac{\alpha}{2}}{\sin\alpha/2}\left\{\sin\frac{3\alpha}{2}\cos\theta - \cos\frac{3\alpha}{2}\sin\theta\right\}$$

i.e. $$x\sin\frac{3\alpha}{2} - y\cos\frac{3\alpha}{2} = a\sin\alpha\cos\frac{\alpha}{2} \quad ...(1)$$

Similarly the equation of the normal at D i.e. at $\theta = \pi + \alpha$ is

$$x\sin\frac{3(\pi+\alpha)}{2} - y\cos\frac{3(\pi+\alpha)}{2} = a\sin(\pi+\alpha)\cos\frac{\pi+\alpha}{2}$$

$$\Rightarrow \quad -x\cos\frac{3\alpha}{2} - y\sin\frac{3\alpha}{2} = a\sin\alpha\sin\frac{\alpha}{2}$$

$$\Rightarrow \quad x\cos\frac{3\alpha}{2} + y\sin\frac{3\alpha}{2} = -a\sin\alpha\sin\frac{\alpha}{2}. \quad ...(2)$$

Squaring (1) and (2) and then adding, we get

$$x^2 + y^2 = a^2\sin^2\alpha. \quad ...(3)$$

Moreover solving (1) and (2), we find $x = a\sin^2\alpha$.

Therefore from (3), $x^2 + y^2 = ax$

which represents the equation of a circle. Hence proved.

Example 21

1. Find the equation of the normal at the point $\theta = \alpha$ on the conic $\frac{l}{r} = 1 + e\cos\theta$.

2. Find the equation of the normal at the point $\theta = 2\alpha$ on the cardiode $r = a(1 + \cos\theta)$. **(P.U. 58H, 60H; M.U. 71H)**

3. Prove that the tangents at the ends of a focal chord of the cardiode $r = a(1 + \cos\theta)$ intersect at right angles on a circle. Find the circle.

(Bhag. 62H, 68H, 72H, P.U. 93H)

4. If the normals of the points (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) on the cardiode $r = a(1 + \cos\theta)$ be concurrent, show that

$$\Sigma\tan\frac{\theta_1}{2} + 3\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2} = 0.$$

(P.U. 56H, B.U. 66H; M.U. 71H)

[Hints: The equation of the normal at $\theta = \alpha$ is

$$x\sin\frac{3\alpha}{2} - y\cos\frac{3\alpha}{2} = a\sin\alpha\cos\frac{\alpha}{2}.$$

If this normal passes through the point (h, k), then

$$h\sin\frac{3\alpha}{2} - k\cos\frac{3\alpha}{2} = a\sin\alpha\cos\frac{\alpha}{2} \quad ...(1)$$

We are going to show that three normals can be drawn from the point (h, k) to the cardiode.

From (1), we have

$$h\left[3\sin\frac{\alpha}{2} - 4\sin^3\frac{\alpha}{2}\right] - k\left[4\cos^3\frac{\alpha}{2} - 3\cos\frac{\alpha}{2}\right] = a\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}$$

which after division by $\cos^3\frac{\alpha}{2}$ becomes

$$h\left[3\tan\frac{\alpha}{2}\sec^2\frac{\alpha}{2} - 4\tan^3\frac{\alpha}{2}\right] - k\left[4 - 3\sec^2\frac{\alpha}{2}\right] = 2a\tan\frac{\alpha}{2}$$

$$\Rightarrow h\left[3\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right) - 4\tan^3\frac{\alpha}{2}\right] - k\left[4 - 3\left(1+\tan^2\frac{\alpha}{2}\right)\right] - 2a\tan\frac{\alpha}{2} = 0$$

$$\Rightarrow \left\{3h\tan\frac{\alpha}{2} - h\tan^3\frac{\alpha}{2}\right\} - \left\{k - 3k\tan^2\frac{\alpha}{2}\right\} - 2a\tan\frac{\alpha}{2} = 0.$$

$$\Rightarrow -h\tan^3\frac{\alpha}{2} + 3k\tan^2\frac{\alpha}{2} + (3h - 2a)\tan\frac{\alpha}{2} - k = 0$$

$$\Rightarrow h\tan^3\frac{\alpha}{2} - 3k\tan^2\frac{\alpha}{2} - (3h - 2a)\tan\frac{\alpha}{2} + k = 0. \quad ...(1)$$

This is a cubic equation in $\tan\frac{\alpha}{2}$; hence three normals will be drawn from (h, k) to the cardiode.

Let the three roots of (1) be $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$ and $\tan\frac{\theta_3}{2}$.

Then $\Sigma\tan\frac{\theta_1}{2} = \frac{3k}{h}$ and $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2} = -\frac{k}{h}$

$$\therefore \quad \Sigma\tan\frac{\theta_1}{2} = -3\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2}\cdot\tan\frac{\theta_3}{2}.$$

Hence thr result.

5. If the normals at three points on the cardiode $r = a(1 + \cos\theta)$ be concurrent, and if the sum of the radii vectors of these points be constant, then show that the locus of the point of concurrence of the normals is a circle passing through the pole. **(Bhag. 67H)**

[Hint : We know from the preceeding example that if the normal at $\theta = \alpha$ on the given curve passes through the point (h, k), then

$$h\tan^3\frac{\alpha}{2} - 3k\tan^2\frac{\alpha}{2} + (2a - 3h)\tan\frac{\alpha}{2} + k = 0 \quad ...(1)$$

This is a cubic equation in $\tan\frac{\alpha}{2}$ and hence three normals, say at $\theta = \theta_1, \theta_2, \theta_3$ pass through (h, k). Thus (h, k) is the point of concurrence of the three normals at $\theta = \theta_1, \theta_2, \theta_3$, satisfying the equation (1).

From (1), $$\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2} = \frac{3k}{h} \quad ...(1)$$

$$\Sigma\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2} = \frac{2a-3h}{h} \quad ...(2)$$

and $$\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2}\cdot\tan\frac{\theta_3}{2} = -\frac{k}{h} \quad ...(3)$$

Given $a(1+\cos\theta_1) + a(1+\cos\theta_2) + a(1+\cos\theta_3) = \text{const.} = l$ (say)

$$\Rightarrow \quad 3a + a(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) = l$$

$$\Rightarrow \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = \frac{l-3a}{a}.$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\theta_1}{2}}{1+\tan^2\frac{\theta_1}{2}} + ... = \frac{l-3a}{a} \quad \Rightarrow \quad \left(\frac{2}{1+\tan^2\frac{\theta}{2}} - 1\right) + ... = \frac{l-3a}{a}$$

$$\Rightarrow \quad \frac{2}{1+\tan^2\theta_1/2} + ... = \frac{l-3a}{a} + 3 = \frac{l}{a}$$

$$\Rightarrow \quad \frac{1}{1+\tan^2\frac{\theta_1}{2}} + \frac{1}{1+\tan^2\frac{\theta_2}{2}} + \frac{1}{1+\tan^2\frac{\theta_3}{2}} = \frac{l}{2a}. \quad ...(4)$$

Thus we have to eliminate $\theta_1, \theta_2, \theta_3$ from (1), (2), (3) and (4).

Squaring (1), we get

$$\left\{\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2}\right\}^2 = \frac{9k^2}{h^2}$$

$$\Rightarrow \quad \Sigma\tan^2\frac{\theta_1}{2} + 2\Sigma\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2} = \frac{9k^2}{h^2}$$

$$\therefore \quad \Sigma\tan^2\frac{\theta_1}{2} = \frac{9k^2}{h^2} - 2\cdot\frac{2a-3h}{h} = \frac{9k^2 - 2h(2a-3h)}{h^2} \quad ...(5)$$

Squaring (2), we get

$$\left\{\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} + \tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2} + \tan\frac{\theta_3}{2}\tan\frac{\theta_1}{2}\right\}^2 = \frac{(2a-3h)^2}{h^2}$$

$$\Rightarrow \quad \Sigma\tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2} + 2\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\left\{\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2}\right\} = \frac{(2a-3h)^2}{h^2}$$

$$\Rightarrow \quad \Sigma\tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2} + 2\left(-\frac{k}{h}\right)\left\{\frac{3k}{h}\right\} = \frac{(2a-3h)^2}{h^2}$$

$$\therefore \quad \Sigma\tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2} = \frac{(2a-3h)^2}{h^2} + \frac{6k^2}{h^2} = \frac{(2a-3h)^2 + 6k^2}{h^2} \qquad ...(6)$$

From (4), L.H.S.

$$\left(1+\tan^2\frac{\theta_2}{2}\right)\left(1+\tan^2\frac{\theta_3}{2}\right) + \left(1+\tan^2\frac{\theta_1}{2}\right)\left(1+\tan^2\frac{\theta_3}{2}\right) + \left(1+\tan^2\frac{\theta_1}{2}\right)\left(1+\tan^2\frac{\theta_2}{2}\right)$$

$$= \frac{}{\left(1+\tan^2\frac{\theta_1}{2}\right)\left(1+\tan^2\frac{\theta_2}{2}\right)\left(1+\tan^2\frac{\theta_3}{2}\right)} \qquad ...(7)$$

The num. $= 3 + 2\,\Sigma\tan^2\frac{\theta_1}{2} + \Sigma\tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2}$

$$= 3 + 2\times\frac{9k^2 - 2h(2a-3h)}{h^2} + \frac{(2a-3h)^2 + 6k^2}{h^2}$$

$$= \frac{1}{h^2}[3h^2 + 2(9k^2 - 4ah + 6h^2) + (4a^2 - 12ah + 9h^2 + 6k^2)]$$

$$= \frac{1}{h^2}[24h^2 + 24k^2 - 20ah + 4a^2].$$

The denom. $= 1 + \Sigma\tan^2\frac{\theta_1}{2} + \Sigma\tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2} + \tan^2\frac{\theta_1}{2}\tan^2\frac{\theta_2}{2}\tan^2\frac{\theta_3}{2}$.

$$= 1 + \frac{9k^2 - 4ah + 6h^2}{h^2} + \frac{4a^2 - 12ah + 9h^2 + 6k^2}{h^2} + \frac{k^2}{h^2}$$

$$= \frac{1}{h^2}[h^2(9k^2 - 4ah + 6h^2) + (4a^2 - 12ah + 9h^2 + 6k^2 + k^2)]$$

$$= \frac{1}{h^2}[16h^2 + 16k^2 - 16ah + 4a^2].$$

Thus (7) reduces to

$$\frac{24h^2 + 24k^2 - 20ah + 4a^2}{16h^2 + 16k^2 - 16ah + 4a^2} = \frac{6h^2 + 6k^2 - 5ah + a^2}{4h^2 + 4k^2 - 4ah + a^2}.$$]

ANSWER

1. $\frac{el\sin\alpha}{1+e\cos\alpha}u = e\sin\alpha\cos(\theta-\alpha) + (1+e\cos\alpha)\sin(\theta-\alpha)$.

2. $r\sin(3\alpha-\theta) = \frac{a}{2}(\sin 3\alpha + \sin\alpha)$ 3. $\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{3a}{2}\right)^2$

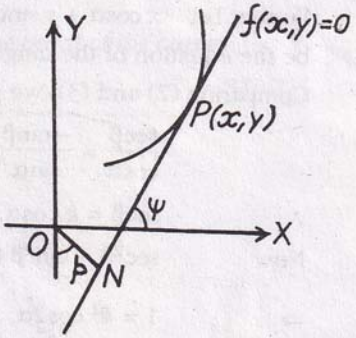
5.17. Pedal Curves

Definition—*If a perpendicular be drawn from a fixed point on a variable tangent to a curve, the locus of the foot of the perpendicular is called* ***the first positive pedal*** *of the original curve with respect to the given point.*

Generally the fixed point is taken to be pole.

5.18. To find the first positive pedal with regard to the origin of any curve whose equation is $f(x, y) = 0$.

Let The equation of the curve be $f(x, y) = 0$.



Let $X\cos\alpha + Y\sin\alpha = p$. ...(1)

be the tangent to the curve at the point (x, y). Let $ON \perp OX$.

Then $ON = p$ and $\angle NOX = \alpha$.

Now, the equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y - y = \left(\frac{dy}{dx}\right)X - x\frac{dy}{dx}$$

$$\Rightarrow \quad Y - \left(\frac{dy}{dx}\right)X = y - x\frac{dy}{dx} \qquad \ldots(2)$$

Comparing (1) and (2), we get

$$\frac{1}{\sin\alpha} = \frac{-\frac{dy}{dx}}{\cos\alpha} = \frac{y - x\frac{dy}{dx}}{p} = k \text{ (say)}$$

$$\Rightarrow \quad \left.\begin{aligned} 1 &= k\sin\alpha \\ -\frac{dy}{dx} &= k\cos\alpha \\ y - x\frac{dy}{dx} &= kp \end{aligned}\right\} \qquad \ldots(3)$$

If we eliminate x, y, k from (1) and (3), we will get a equation in p and α only. Here p and α are the polar co- ordinates of the foot N of the perpendicular. Hence by writing r in place of p and θ in place of α we shall get the polar equation of the locus of N and consequently the equation of the first positive pedal of the given curve.

EXAMPLE 1. *Find the first positive pedal of the curve* $x^2 - y^2 = a^2$.

SOLUTION : Given $x^2 - y^2 = a^2$ ...(1)

Let $P(x_1, y_1) = (a\sec\beta, a\tan\beta)$ be any point on the curve.

Then the equation of the tangent at this point is $xx_1 - yy_1 = a^2$

i.e. $x a\sec\beta - y a\tan\beta = a^2$

$\Rightarrow$ $x\sec\beta - y\tan\beta = a$ ...(2)

Further Let $x\cos\alpha + y\sin\alpha = p$ ...(3)

be the equation of the tangent to the curve at the point P.

Comparing (2) and (3) ,we get

$$\frac{\sec\beta}{\cos\alpha} = \frac{-\tan\beta}{\sin\alpha} = \frac{a}{p} = k \text{ (say).}$$

$\therefore$ $\sec\beta = k\cos\alpha$, $-\tan\beta = k\sin\alpha$; $a = pk$

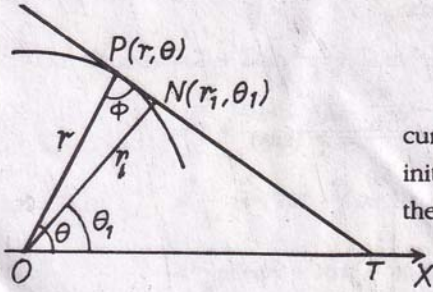
Now $\sec^2\beta - \tan^2\beta = k^2(\cos^2\alpha - \sin^2\alpha)$

$\Rightarrow$ $1 = k^2\cos 2\alpha \Rightarrow 1 = \frac{a^2}{p^2}\cos 2\alpha$

$\therefore$ $p^2 = a^2\cos 2\alpha$.

Hence the equation of the first positive pedal is $r^2 = a^2\cos 2\theta$.

5.19. To find the pedal with regard to the pole of any curve whose polar equation is $f(r, \theta) = 0$.



Here the equation of the curve is

$$f(r,\theta) = 0 \quad \text{...(1)}$$

Let $P(r,\theta)$ be any point on the curve and let the tangent at P meet the initial line at T. Draw $ON \perp PT$ and let the co-ordinates of N be (r_1, θ_1).

It is evident from the figure that

$$\theta = \angle XOP = \angle XON + \angle NOP = \theta_1 + \left(\frac{\pi}{2} - \phi\right)$$

i.e. $\theta = \theta_1 + \frac{\pi}{2} - \phi$ ...(2)

Also $\tan\phi = \frac{r d\theta}{dr}$ ..

Again, from ΔPON, $\sin\phi = \frac{ON}{OP} = \frac{r_1}{r}$

$\therefore$ $r_1 = r \sin \phi$

and $\frac{1}{r_1^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ ..

Eliminating r, θ, ϕ from (1), (2), (3) and (4) we will get an equation c necting r_1 and θ_1. Writing r in place of r_1 and θ in place of θ_1, we shall the required pedal with respect to pole.

EXAMPLE 2. *Find the equation of the first positive pedal of the curve*

$$r^m = a^m \cos m\theta.$$ **(B.U. 72**

SOLUTION : Given $r^m = a^m \cos m\theta$ ...

Taking log on both sides,

$$m \log r = m \log a + \log\cos m\theta.$$

Differentiating w.r.t. θ, we get

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{-m \sin m\theta}{\cos m\theta}$$

$\Rightarrow$ $\frac{r d\theta}{dr} = -\cot m\theta$

$\Rightarrow$ $\tan\phi = \tan\left(\frac{\pi}{2} + m\theta\right)$ $\therefore$ $\phi = \frac{\pi}{2} + m\theta$ ...(

Again $\theta = \theta_1 + \frac{\pi}{2} - \phi = \theta_1 + \frac{\pi}{2} - \frac{\pi}{2} - m\theta$

$\Rightarrow$ $(m+1)\theta = \theta_1$ $\therefore$ $\theta = \frac{\theta_1}{m+1}$ ...(

Also $r_1 == r\sin\phi \Rightarrow r_1 = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta$ ...(

Now, from (4), $r_1 = r\cos m\theta$

$\Rightarrow$ $r_1^m = r^m \cos^m m\theta$

$= a^m \cos m\theta . \cos^m m\theta$

$= a^m \cos^{m+1}(m\theta)$

$$= a^m \cos^{m+1}\left(\frac{m}{m+1}\theta_1\right); \text{ from (3)}$$

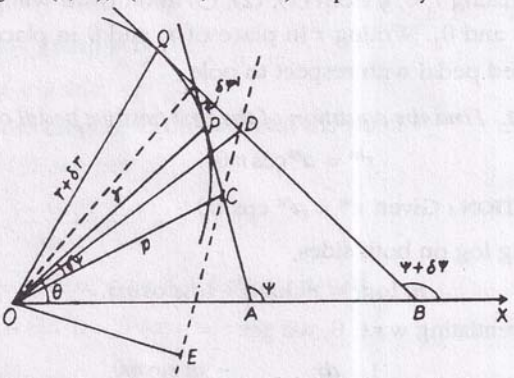
$$\Rightarrow \quad r_1^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos\left(\frac{m}{m+1}\theta_1\right).$$

Hence the equation of the pedal curve is

$$r^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos\left(\frac{m}{m+1}\theta\right).$$

5.20. Perpendicular on tangent to pedal.

Let P and Q be two contiguous points on the curve and let the co-ordinates of P and Q be (r, θ), $(r + \delta r, \theta + \delta\theta)$ respectively.



Let PA and QB be two tangents at P and Q respectively which intersect at T.

Let the tangents make angles ψ and $\psi + \delta\psi$ with the initial line.

Draw $OC \perp TA$ and $OD \perp TB$.

The angle between the two tangents is $\delta\psi$. Therefore the angle between their two normals is $= \delta\psi$; i.e. $\angle COD = \delta\psi$.

Draw $OE \perp DC$ produced.

Let $OC = p$ and $OD = p + \delta p$.

Since $\angle CTD = \angle COD = \delta\psi$, therefore the points O, C, D, T are concyclic.

Hence ext. $\angle OCE = \angle OTD$.

Now $\Delta^s OCE$ and OTD are similar ($\because$ they are equiangular)

$$\therefore \quad \frac{OE}{OC} = \frac{OD}{OT}. \qquad \ldots(1)$$

Now, take the limit as Q coincides with P. In that case, $D \to C$ and in the limit DC becomes the tangent to the pedal curve. Let the length of perpendicular on the tangent at C on the pedal curve be p_1. Then (1) becomes

$$\frac{p_1}{p} = \frac{p}{r} \quad \text{i.e. } p_1 r = p^2 \qquad ...(2)$$

Cor. We have $\angle OCE = \angle OTD$. $\therefore$ when Q tends to coincide with P and consequently when DC becomes tangent to the pedal curve, $\angle OPA$ tends to become $\angle OCE$. Thus if ϕ and ϕ' refer to the original curve and pedal curve respectively, then $\phi = \phi'$.

5.21. Pedal equation of pedal curve.

Let the pedal equation of the given curve be $r = f(p)$ ...(1)

We have seen in the preceding article, that $p_1 r = p^2 \therefore r = \frac{p^2}{p_1}$ ...(2)

Putting the value of r in (1), we have

$$\frac{p^2}{p_1} = f(p) \Rightarrow p_1 = \frac{p^2}{f(p)} \qquad ...(3)$$

Therefore writing r for p and p for p_1, the required pedal equation of the first positive pedal curve is $p = \frac{r^2}{f(r)}$.

EXAMPLE *Show that the first positive pedal of the curve* $p = \frac{r^3}{a^2}$ *is* $p^3a^2 = r^5$.

SOLUTION : Let p_1, r_1 refer to the pedal so that $p_1 r = p^2$ or $r = \frac{p^2}{p_1}$.

$\therefore$ Putting the value of r in the given curve, we get

$$p = \frac{p^6}{a^2 p_1^3} \Rightarrow a^2 p_1^3 = p^5.$$

Therefore, writing r for p and p for p_1, the required pedal equation of the first positive pedal of the given curve is $p^3a^2 = r^5$.

5.22. Different orders of positive or negative pedal curves.

Let there be a series of curves viz, F, F_1, F_2, F_3, ... F_n, F_{n+1}, ...

Let us assume that each of the above curves is the first positive pedal of the one just preceding it; that is, F_1 is first positive pedal of F, F_2 is the first positive pedal of F_1 and so on. Under these conditions, F_2 is called the second positive pedal of F, F_3 is called the third positive pedal of F and in this case F_k is called the kth positive pedal of F and so on. Further, if any one of

the above curves may be treated as the original curve, say F_3, then F_2 is called the first negative pedal of F_3 and F_1 is the second negative pedal of F_3 and so on.

EXAMPLE 1. *Find the kth positive pedal of* $r^n = a^n \cos n\theta$.

SOLUTION : We have seen in Ex. 2 on page 257 that the first positive pedal of $r^n = a^n \cos n\theta$ is $r^{n_1} = a^{n_1} \cos n_1\theta$ where $n_1 = \dfrac{n}{n+1}$.

Similarly, the 2nd positive pedal is

$$r^{n_2} = a^{n_2} \cos n_2\theta \text{ where } n_2 = \frac{n_1}{n_1 + 1} = \frac{\frac{n}{n+1}}{\frac{n}{n+1} + 1} = \frac{n}{1 + 2n}.$$

Proceeding in this way, we find that the k th positive pedal is

$$r^{n_k} = a^{n_k} \cos n_k\theta \text{ where } n_k = \frac{n}{1 + kn}.$$

That is, the k th positive pedal is of the same form $r^m = a^m \cos m\theta$ where $m = \dfrac{n}{1 + kn}$.

EXAMPLE 2. *Find the 4th negative pedal of the curve* $r^m = a^m \cos m\theta$.

SOLUTION : Let the 4th negative pedal of the given curve is $r^k = a^k \cos k\theta$.

Then $r^m = a^m \cos m\theta$ is the 4th positive pedal of $r^k = a^k \cos k\theta$.

$\therefore$ From the preceding example, $m = \dfrac{k}{1 + 4k}$

i.e. $k = m + 4mk \Rightarrow k(1 - 4m) = m \quad \therefore \quad k = \dfrac{m}{1 - 4m}$.

Hence the 4th negative pedal of the given curve is $r^k = a^k \cos k\theta$ where $k = \dfrac{m}{1 - 4m}$.

Example 22

1. Find the kth positive pedal of $r^m = a^m \cos m\theta$.
2. Find the kth negative pedal of $r^m = a^m \cos m\theta$. **(M.U. 70H, 72H, 74H)**
3. Find the 4th positive pedal of the curve $r^2 = a^2 \cos 2\theta$.
4. Find the first positive pedal of the curve $Ax^m + By^m = 1$.
5. Find the first negative pedal of the parabola $y^2 = 4ax$ w.r.t. the vertex.

6. Show that the first positive pedal of the curve $p = \frac{r^{m+1}}{a^m}$ is $p^{m+1}a^m = r^{2m+1}$.

ANSWER

1. $r^n = a^n \cos n\theta$ where $n = \frac{m}{1+km}$

2. $r^n = a^n \cos n\theta$ where $n = \frac{m}{1-km}$

3. $r^n = a^n \cos n\theta$ where $n = \frac{2}{9}$.

4. $\frac{m}{r^{m-1}} = \frac{\cos^{\frac{m}{1}-1}\theta}{A^{\frac{m}{1}-1}} + \frac{\sin^{\frac{m}{1}-1}\theta}{B^{\frac{m}{1}-1}}$.

5. $7ay^2 = (x-4a)^3$.

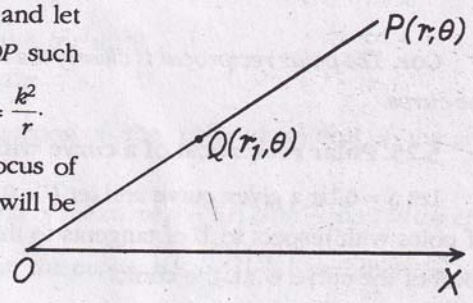
5.23. Inversion

Definition—*Let P be any given point and let there be any other point Q on OP, where O is the pole such that $OP.OQ = k^2$ (say), then Q is called the inverse of the point P with regard to a circle of radius k and centre O.*

Further if the point moves in any assigned manner, the locus of Q is called the inverse to the path of P.

I. Polar Co-ordinates

Let $P(r, \theta)$ be any given point and let $Q(r_1, \theta_1)$ be any other point on OP such that $OP.OQ = k^2$ or $r\,r_1 = k^2 \quad \therefore\ r_1 = \frac{k^2}{r}$.



Thus if we suppose that the locus of P is $f(r, \theta) = 0$, then the locus of Q will be $f\left(\frac{k^2}{r}, \theta\right) = 0$.

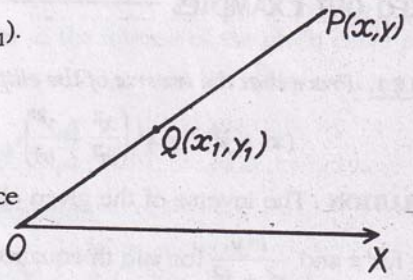
II. Cartesian Co-ordinates

Let the cartesian co-ordinates of P be (x, y) and those of Q be (x_1, y_1).

Now, $x = r\cos\theta$, $y = r\sin\theta$;

$x_1 = r_1\cos\theta$, $y_1 = r_1\sin\theta$.

$\therefore \quad x = r\cos\theta = \frac{k^2}{r_1}\cos\theta$, since $r_1 = k^2$



$$= \frac{k^2 r_1}{r_1^2}\cos\theta = \frac{k^2(r_1\cos\theta)}{r_1^2} = \frac{k^2 x_1}{x_1^2 + y_1^2}.$$

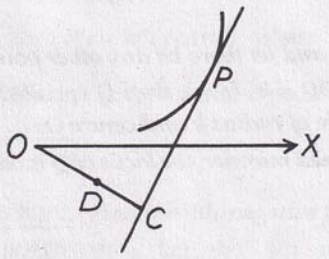
Similarly $y = \dfrac{k^2 y_1}{x_1^2 + y_1^2}$.

Thus if the locus of P be given to be $f(x, y) = 0$, then we shall have

$$f\left(\frac{k^2 x_1}{x_1^2 + y_1^2}, \frac{k^2 y_1}{x_1^2 + y_1^2}\right).$$

That is, the locus of Q will be $f\left(\dfrac{k^2 x}{x^2 + y^2}, \dfrac{k^2 y}{x^2 + y^2}\right) = 0$.

5.24. Polar reciprocal of a curve with regard to a given circle.



Let OC be the perpendicular drawn from the pole (origin) upon the tangent to a given curve. Let D be a point in OC (or OC produced) such that $OC.OD = k^2$.

Then the locus of D is called the polar reciprocal of the given curve with respect to a circle of radius k and centre at O.

Cor. *The polar reciprocal is clearly the inverse of the first positive pedal of the curve.*

5.25. Polar reciprocal of a curve with regard to a given conic.

Let $S = 0$ be a given curve and let $U = 0$ be a given conic. Then the locus of poles with respect to U of tangents to the curve is said to be polar reciprocal of the curve w.r.t. the conic.

Worked out Examples

EXAMPLE 1. *Prove that the inverse of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *w.r.t. the origin is*

$$(x^2 + y^2)^2 = k^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right). \qquad \textbf{(B.U. 65H)}$$

SOLUTION : The inverse of the given ellipse will be obtained by putting $\dfrac{k^2 x}{x^2 + y^2}$ for x and $\dfrac{k^2 y}{x^2 + y^2}$ for y in th equation of the ellipse.

Thus the reqd. equation is

$$\frac{1}{a^2}\left\{\frac{k^2x}{x^2+y^2}\right\}^2+\frac{1}{b^2}\left\{\frac{k^2y}{x^2+y^2}\right\}^2=1$$

$$\Rightarrow \quad \frac{1}{a^2}\frac{k^4x^2}{(x^2+y^2)^2}+\frac{1}{b^2}\frac{k^4y^2}{(x^2+y)^2}=1$$

$$\Rightarrow \quad k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=(x^2+y^2)^2.$$

EXAMPLE 2. *Find the polar reciprocal of an ellipse with regard to its centre.*

SOLUTION : Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $x\cos\alpha + y\sin\alpha = p$ be the tangent to the ellipse. Then, as we have

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.$$

Hence the polar equation of the pedal with respect to the origin is

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

Therefore the inverse of this curve is

$$\left(\frac{k^2}{r}\right)^2 = a^2\cos^2\theta + b^2\sin^2\theta \quad (\because\ rr_1 = k^2)$$

$$\Rightarrow \quad k^4 = a^2r^2\cos^2\theta + b^2r^2\sin^2\theta$$

$$= a^2x^2 + b^2y^2$$

which is the required equation of the polar reciprocal of the given ellipse.

EXAMPLE 3. *Prove that the inverse of the curve $p = f(r)$ with respect to the circle whose radius is a and centre at the pole is $p = \frac{r^2}{a^2}f\left(\frac{a^2}{r}\right)$ and that the polar reciprocal is $\frac{a^2}{r} = f\left(\frac{a^2}{p}\right)$.* **(P.U. 47H; B.U. 63H; M.U. 64H; 69H; 72H)**

SOLUTION : Let dashes refer to the inverse of the given curve $p = f(r)$.

Thus $rr' = a^2$ and $\phi = \phi'$.

Hence $p = r\sin\phi = \frac{a^2}{r'}\sin\phi' = \frac{a^2}{r'^2}r'\sin\phi' = \frac{a^2p'}{r'^2}$.

Therefore from the given equation,

$$\frac{a^2p'}{r'^2} = f\left(\frac{a^2}{r'}\right) \Rightarrow p' = \frac{r'^2}{a^2}f\left(\frac{a^2}{r'}\right).$$

Hence the inverse is given by $p = \frac{r^2}{a^2} f\left(\frac{a^2}{r}\right)$.

This proves the first part.

For the second part, we shall first of all find the first positive pedal of $p = f(r)$ and then its inverse.

For this, we have $p_1 r = p^2 \Rightarrow r = \frac{p^2}{p_1}$.

Putting the value of r in the given equation, we get

$$p = f\left(\frac{p^2}{p_1}\right).$$

Hence the first positive pedal of the given curve, by wiriting r for p and p for p_1 is

$$r = f\left(\frac{r^2}{p}\right). \qquad ...(1)$$

Next, as we know from the first part, in order to find the inverse of (1), we shall have to replace p by $\frac{a^2 p}{r^2}$ and r by $\frac{a^2}{r}$.

Thus the polar equation, from (1), is

$$\frac{a^2}{r} = f\left(\frac{a^4}{r^2} \cdot \frac{r^2}{a^2 p}\right) \quad \text{i.e.} \quad \frac{a^2}{r} = f\left(\frac{a^2}{p}\right).$$

Note : This is an example in which we first of all find the first positive pedal and then its inverse. In the following example, we shall at first find the inverse of a given curve given in pedal equation and then its pedal.

EXAMPLE 4. *Show that the pedal of the inverse of $p = f(r)$ w.r.t. a circle whose radius is k and the centre at the origin is,*

$$k^2 p^2 = r^3 f\left(\frac{k^2 p}{r^2}\right)$$ **(P.U. 57H, 63H; B.U. 62H; Bhag 63H, 67H)**

SOLUTION : For finding out the inverse, we shall have to write $\frac{k^2 p}{r^2}$ for p and $\frac{k^2}{r}$ for r in the given pedal equation of the curve.

Hence the inverse of the given pedal equation is

$$\frac{k^2 p}{r^2} = f\left(\frac{k^2}{r}\right). \qquad ...(1)$$

For finding out the pedal, we have $p_1 r = p^2$ i. e. $r = \frac{p^2}{p_1}$.

Hence from (1), by substituting the value of r,

$$\frac{k^2 p}{p^4} \cdot p_1^{\,2} = f\left(\frac{k^2 p_1}{p^2}\right).$$

Therefore writing r for p and p for p_1, the required pedal equation is

$$\frac{k^2}{r^3} p^2 = f\left(\frac{k^2 p}{r^2}\right) \Rightarrow k^2 p^2 = r^3 f\left(\frac{k^2 p}{r^2}\right).$$

Example 23

1. Find the inverse of the normal at the point $\theta = 2\alpha$ on the cardiode $r = a(1 + \cos\theta)$ w.r.t. a circle of radius a and centre at the origin.

(P.U. 58H, 60H, M.U. 71H)

2. Show that the polar reciprocal of the curve $p = \frac{r^{m+1}}{a^m}$ with regard to a circle of radius a whose centre is at the origin is $p^{m+1} = a^m r$.

(M.U. 79H)

3. Show that the pedal of the inverse of $p = \frac{r^{m+1}}{a^m}$ with regard to a circle whose radius is k and centre at the origin is

$$p = \left(\frac{a}{k^2}\right)^{\frac{m}{m-1}} \cdot \frac{2m-1}{r^{m-1}}$$ **(P.U. 46H, 60H; B.U. 57H, 67H; Bhag. 73H)**

[**Hints:** For inverse, we have to write $\frac{k^2 p}{r^2}$ for p and $\frac{k^2}{r}$ for r in the equation of the curve. With these substitutions, the inverse of the given curve is

$$\frac{k^2 p}{r^2} = \frac{(k^2/r)^{m+1}}{a^m} = \frac{k^{2m+2}}{r^{m+1} a^m}.$$

For the first positive pedal, we have the equation $p = r^2$ i.e. $r = \frac{p^2}{p_1}$.

Putting this value of r in (1), we get

$$\frac{k^2 p p_1^{\,2}}{p^4} = \frac{k^{2m+2} p_1^{m+1}}{a^m \cdot p^{2m+2}}$$

At last, writing r for p and p for p_1, the reqd. equation is

$$\frac{k^2.p^2}{r^3} = \frac{k^{2m+2}.p^{m+1}}{a^m.r^{2m+2}}$$

$$\Rightarrow \qquad p^{m-1} = \frac{a^m.r^{2m-1}}{k^{2m}}.$$ Hence the result.]

4. Show that the polar reciprocal of the curve $r^m = a^m \cos m\theta$ w.r.t. a circle having the pole as centre is of the form

$$r^{\frac{m}{m+1}} \cos^{\frac{m}{m+1}}\theta = b^{\frac{m}{m+1}}.$$

(P.U. 45H, 50H; B.U. 56H; R.U. 69H; M.U. 79H)

[**Hint :** The first positive pedal of the given curve is

$$r^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos^{\frac{m}{m+1}}\theta.$$

(see Ex. 2 worked out on page 257)

For inverse, replace r by $\frac{k^2}{r}$ so as to get the polar reciprocal of the curve]

5. Find the polar reciprocal of the curve $r^m = a^m \cos m\theta$ w.r.t. the hyperbola $r^2 = a^2 \sec 2\theta$. **(P.U. 48H, Bhag. 66H; M.U. 66H, 73H)**

[Use $rr_1 = a^2$.]

ANSWER

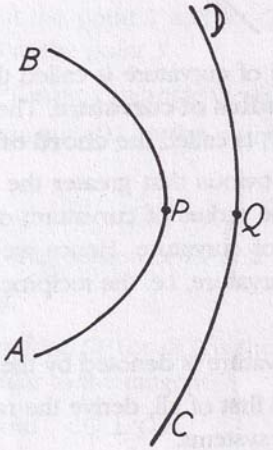
1. $r\sin(3\alpha - \theta) = \frac{a}{2}(\sin 3\alpha + \sin\alpha)$.

5. $r^n \cos n\theta = a^n$ where $n = \frac{m}{m+1}$.

6

CURVATURE

6.1. Introduction : In common parlance, we often speak that a particular road has a small bend or that there are sharp ends ina particular hilly road etc. In mathematics, when we use the word 'curvature' it also means 'bend'. For example, We take two curves AB and CD as in the adjoining figure. From

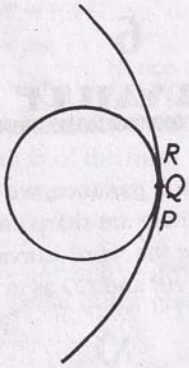


the figure, it is obvious that the curve AB has a larger bend at the point P than that of the curve CD at the point Q. It means, geometricaly, theat the rate of change of the tangent, as it passes through P, is fast as compared to that at the point Q. Thus the bend is determined by the rate of change of direction of the tangent. It has to be remembered that when we consider the curvature of any curve, it will be at a point oc the curve.

Now the problem is how to measure the curvature of a curve at a given point.

Definition—Let P be any point on the curve. We take two points Q and R very near to P.

We know that one and only one circle can pass through three points P, Q, R.. Now we tend the points Q and R to coincide with P. By so doing there will be a limiting position of the circle PQR. This circle is called the **circle of curvature**. Hence when Q, $R \to P$, then the limiting position of the circle PQR is called the circle of curvature.



The centre of this circle of curvature is called the **centre of curvature** and its radius is called the **radius of curvature**. The chord drawn inside the circle of curvature through P, is called the **chord of curvature**.

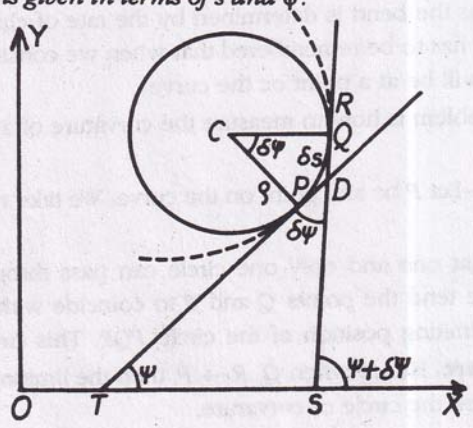
From the figure, it is obvious that greater the bend of the curve at a point, the smaller will be the radius of curvature; or lesser the bend of the curve, greater is the radius of curvature. Hence we define curvature as the reciprocal of the radius of curvature, i.e. the reciprocal of the radius of curvature is called the curvature.

Generally, radius of curvature is denoted by the symbol ρ.

In this chapter, we shall first of all, derive the radius of curvature in the case of different co-ordinate systems.

6.2. Formula for Radius of curvature

***I. Intrinsic Formula**—To find the radius of curvature for the curve whose equation is given in terms of s and ψ.*



Let P, Q, R be any three points on the curve. Draw a circle through all the three points P, Q, R. Join PQ and QR. Then PQ and QR are the chords (secant) of the circle PQR.

Now we coincide R with Q and Q with P. When $R \to Q$ and $Q \to P$, the three points P, Q, R become coincident and the circle PQR transforms into circle of curvature.

Let C be the centre of this circle of curvature.

Again, when $Q \to P$, the secant PQ becomes the tangent at the point P and when $R \to Q$, the secant QR becomes the tangent at the point Q.

These cases have been illustrated in the figure above. Let PT be a tangent at P which meets x-axis at the point T and let QS be a tangent at the point Q which meets the x-axis at the point S.

Let us suppose that PT makes an angle ψ with the x-axis. Since Q is very near to P, therefore the tangent QS, at the point Q makes an angle $\psi + \delta\psi$ with the x-axis.

i.e. $\angle PTX = \psi$ and $\angle QSX = \psi + \delta\psi$.

Let D be the point of intersection of PT and QS.

Therefore $\angle TDS = \delta\psi$.

Join PC and QC. Therefore CP (or CQ) is the radius of curvature ρ. Since radius is perpendicular to the tangent

$\therefore$ $CP \perp PT$ and $CQ \perp QS$.

Therefore $\angle PCQ = 180° - \angle PDQ = 180° - (180° - \delta\psi)$

$= \delta\psi$.

Let the arcual distance of P from any fixed point A be s.

i.e. arc $AP = s$ $\quad\therefore\quad$ arc $AQ = s + \delta s$

$\therefore$ arc $PQ = \delta s$.

Now in the circle of curvature, by applying the formula, $\theta = \frac{l}{r}$, we have

$$\delta\psi = \frac{\text{arc } PQ}{\rho} \quad \text{when} \quad Q \to P.$$

$$\rho = \underset{Q \to P}{Lt} \frac{\text{arc } PQ}{\delta\psi} = \underset{\delta\psi \to 0}{Lt} \frac{\delta s}{\delta\psi} = \frac{ds}{d\psi}.$$

Note : ***radius of curvature of a circle.***

Let the radius of the circle = a.

Then, as is evident from the fig.

$$\delta\psi = \frac{\delta s}{a} \quad \therefore \quad \frac{ds}{\delta\psi} = a \quad \therefore \quad \rho = a.$$

Hence in a circle, the radius of curvature is constant.

EXAMPLE *Find the radius of curvature at the point* (s, ψ) *of the curve* $s = c\tan\psi$.

SOLUTION : Since $s = c\tan\psi \quad \therefore \quad \frac{ds}{d\psi} = c\sec^2\psi$

$$\therefore \qquad \rho = c\sec^2\psi.$$

II. Cartesian Formula—*To find the radius of curvature for the cartesian curve* $y = f(x)$.

Let the equation of the curve in cartesian form be $y = f(x)$.

We know that $\tan\psi = \frac{dy}{dx}$.

Differentiating this w.r.t. x, we get

$$\sec^2\psi \frac{d\psi}{dx} = \frac{d^2y}{dx^2} \Rightarrow \sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \sec^2\psi \cdot \frac{1}{\rho} \cdot \frac{1}{\cos\psi} = \frac{d^2y}{dx^2} \Rightarrow \frac{1}{\rho} \cdot \sec^3\psi = \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{1}{\rho}(1 + \tan^2\psi)^{3/2} = \frac{d^2y}{dx^2} \Rightarrow \frac{1}{\rho}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \frac{d^2y}{dx^2}$$

$$\therefore \quad \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{where} \quad \frac{d^2y}{dx^2} \neq 0.$$

In short, we write this formula as follows:

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}.$$

Failure of the formula.

If y_1 be infinite at any point of the curve i.e. the tangent at that point is parallel to the y-axis, then this formula will fail. In that case, we start with the reciprocal of $\tan\psi = \frac{dy}{dx}$ i.e. we start with $\cot\psi = \frac{dx}{dy}$. The subsequent method is quite the same. We have

$$\cot\psi = \frac{dx}{dy}.$$

Differentiating this w.r.t. y, we get

$$-\operatorname{cosec}^2\psi \cdot \frac{d\psi}{dy} = \frac{d^2x}{dy^2}$$

$$\Rightarrow \quad -\operatorname{cosec}^2\psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dy} = \frac{d^2x}{dy^2} \quad \Rightarrow \quad -\operatorname{cosec}^2\psi \cdot \frac{1}{\rho}\frac{1}{\sin\psi} = \frac{d^2x}{dy^2}$$

$$\Rightarrow \quad -\operatorname{cosec}^3\psi \cdot \frac{1}{\rho} = \frac{d^2x}{dy^2} \quad \Rightarrow \quad -(1+\cot^2\psi)^{3/2} \cdot \frac{1}{\rho} = \frac{d^2x}{dy^2}$$

$$\therefore \quad -\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2} \cdot \frac{1}{\rho} = \frac{d^2x}{d^2y}$$

$$\therefore \quad \text{(in length) } \rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} \quad \text{where } \frac{d^2x}{dy^2} \neq 0.$$

III. Parametric Formula

Let the equation of the curve be $x = f(t)$, $y = g(t)$.

Then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx} = \dfrac{d}{dt}\left(\dfrac{g'(t)}{f'(t)}\right) \cdot \dfrac{1}{\frac{dx}{dt}}$

$$= \frac{f'(t).g''(t) - g'(t)f''(t)}{\{f'(t)\}^2} \cdot \frac{1}{f'(t)}.$$

If we put $f'(t) = x'$, $g'(t) = y'$ ....,

then $\dfrac{dy}{dx} = \dfrac{y'}{x'}$ and $\dfrac{d^2y}{dx^2} = \dfrac{x'y' - y'x''}{(x')^3}$

$\therefore$ From the formula $\rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \dfrac{\left\{1+\left(\frac{y'}{x'}\right)^2\right\}^{3/2}}{\frac{x'y' - y'x''}{(x')^3}}$

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'^3} \times \frac{x'^3}{x'y' - y'x''}$$

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}.$$

Worked out Examples

EXAMPLE 1. *For the curve in which* $s = ce^{x/c}$, *show that* $c\rho = s\sqrt{s^2 - c^2}$.

(P.U. 37A, 64S, 70S, 75A; B.U. 74A)

SOLUTION : Given that $x = ce^{x/c}$.

$$\therefore \quad \frac{ds}{dx} = ce^{x/c} \times \frac{1}{c} = \frac{s}{c}$$

i.e. $\sec\psi = \frac{s}{c} \quad \therefore \quad s = c\sec\psi.$

Differentiating w.r.t. ψ, we get

$$\frac{ds}{d\psi} = c\sec\psi\tan\psi = c\sec\psi\sqrt{\sec^2\psi - 1}$$

$$= s\sqrt{\frac{s^2}{c^2} - 1} = \frac{s}{c}\sqrt{s^2 - c^2}$$

$$\Rightarrow \quad \rho = \frac{s}{c}\sqrt{s^2 - c^2}$$

$$\therefore \quad c\rho = s\sqrt{s^2 - c^2}.$$

MPLE 2. *For any curve, prove the formula*

$$\rho = \frac{r}{\sin\phi\left(1 + \frac{d\phi}{d\theta}\right)} \quad \textit{where} \tan\phi = r\frac{d\theta}{dr}.$$

(P.U. 52A, 62A, 64S; B.U. 54A, 63A, 80A; R.U. 63A, 84; Bhag.U. 65A; M.U. 69S)

SOLUTION : We know that

$$\psi = \theta + \phi \quad \therefore \quad \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \quad \frac{d\psi}{ds} \cdot \frac{ds}{d\theta} = 1 + \frac{d\phi}{d\theta} \quad \Rightarrow \quad \frac{1}{\rho} \cdot \frac{ds}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \quad \frac{1}{\rho} \cdot \frac{ds}{rd\theta} \cdot r = 1 + \frac{d\phi}{d\theta} \quad \Rightarrow \quad \frac{1}{\rho} \cdot \frac{1}{\sin\phi} \cdot r = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \quad \rho = \frac{r}{\sin\phi\left(1 + \frac{d\phi}{d\theta}\right)}.$$

EXAMPLE 3. *Find the radius of curvature at any point of the parabola $y^2 = 4ax$ and hence deduce that the radius of curvature at the vertex of a parabola is equal to its semi-latus rectum.* **(P.U. 32A)**

SOLUTION : From the formula,

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}. \qquad ...(1)$$

Given : The equation of the parabola is $y^2 = 4ax$.

$$\therefore \quad 2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{i.e.} \quad y_1 = \frac{2a}{y}.$$

Again, differentiating, we get

$$y_2 = -\frac{2a}{y^2}y_1 = -\frac{2a}{y^2}\cdot\frac{2a}{y} = \frac{-4a^2}{y^3}.$$

$$\therefore \text{ From (1), } \rho = -\frac{\left(1 + \frac{4a^2}{y^2}\right)^{3/2}}{\frac{4a^2}{y^3}} = -\frac{(4a^2 + y^2)^{3/2}}{4a^2}$$

$$= -\frac{(4a^2 + 4ax)^{3/2}}{4a^2} = -\frac{(4a)^{3/2}(a + x)^{3/2}}{4a^2}$$

$$= -\frac{8a^{3/2}(a + x)^{3/2}}{4a^2} = -2a^{-1/2}(a + x)^{3/2}$$

i.e. (in magnitude) $\rho = 2a^{-1/2}(a + x)^{3/2}$. ...(2)

This proves part (i).

For the second part, i.e. for the radius of curvature at the vertex (0, 0) we shall put $x = 0$, $y = 0$ in (2).

Hence at the vertex, $\rho = 2a^{-1/2}a^{3/2} = 2a = \frac{4a}{2} = \frac{1}{2}$ (latus rectum).

Hence the result.

Note : $\rho^2 = 4a^{-1}(a + x)^3 = \frac{4(a + x)^3}{a}$.

Again since the co-ordinates of S are $(a, 0)$ and those of P are (x, y),

$$\therefore \quad SP^2 = (a - x)^2 + y^2 = a^2 - 2ax + x^2 + 4ax$$

$$= a^2 + 2ax + x^2 = (a + x)^2.$$

$$\therefore \quad SP^3 = (a + x)^3.$$

$$\therefore \quad \rho^2 = \frac{4}{a}SP^3 \quad \text{i.e.} \quad \rho^2 \propto (SP)^3.$$

EXAMPLE 4. *Find the radius of curvature at any point* $(a\cos t, b\sin t)$ *of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \textbf{(Bhag. 64A)}$$

SOLUTION : Let the co-ordinates of any point P on the ellipse be $x = a\cos t,\ y = b\sin t.$

$$\therefore \quad \frac{dx}{dt}=-a\sin t;\ \frac{dy}{dt}=b\cos t;$$

$$\frac{d^2x}{dt^2}=-a\cos t;\ \frac{d^2y}{dt^2}=-b\sin t.$$

$\therefore$ From the formula, $\rho=\dfrac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}.$

$$=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab\sin^2 t+ab\cos^2 t}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}.$$

EXAMPLE 5. *In the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *prove that*

$$\rho=\frac{a^2b^2}{p^3}$$ *where* ρ = *radius of curvature.*

p = *central perpendicular on the tangent.* **(P.U. 65A)**

SOLUTION : Let $P\,(a\cos t,\ b\sin t)$ be any point on the ellipse. Then we know from the previous Ex. that

$$\rho=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab} \qquad ...(1)$$

Now the equation of the tangent at the point $P\,(a\cos t,\ b\sin t)$ is

$$\frac{x.a\cos t}{a^2}+\frac{y.b\sin t}{b^2}=1$$

$$\Rightarrow \quad \frac{x\cos t}{a}+\frac{y\sin t}{b}=1.$$

Now, the length of the perpendicular on the tangent from the centre (0, 0),

$$p=\frac{1}{\sqrt{\dfrac{\cos^2 t}{a^2}+\dfrac{\sin^2 t}{b^2}}}=\frac{ab}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}$$

$$\therefore \quad p^2=\frac{a^2b^2}{a^2\sin^2 t+b^2\cos^2 t}.$$

Now, from (1), $\rho=\left(\dfrac{a^2b^2}{p^2}\right)^{3/2}\dfrac{1}{ab}=\dfrac{a^3b^3}{p^3.ab}=\dfrac{a^2b^2}{p^3}.$

Hence the result.

EXAMPLE 6. *If CP and CD be a pair of conjugate semi-diameters of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *prove that the radius of curvature at P is* $\frac{CD^3}{ab}$. **(P.U. 34H, 60A)**

SOLUTION : Let the co-ordinates of P be $(a\cos t, b\sin t)$. We know from Geometry that the co-ordinates of D are

$$\left\{a\cos\left(\frac{\pi}{2}+t\right),\ b\sin\left(\frac{\pi}{2}+t\right)\right\}$$

i.e. $(-a\sin t,\ b\cos t)$.

We know that at the point P,

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \qquad ...(1)$$

Now $CD^2 = (-a\sin t)^2 + (b\cos t)^2 = a^2\sin^2 t + b^2\cos^2 t$

$\therefore$ from (1), $\rho = \frac{CD^3}{ab}$.

EXAMPLE 7. *If* ρ *and* ρ' *be the radii of curvature at the extremities of two conjugate diameters of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *prove that*

$$(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2.$$

(P.U. 33A, 38H, 59A; M.U. 63A, Bhag. 91H)

SOLUTION : Let CP and CD be two semi-conjugate diameters of the ellipse.

Let the co-ordinates of P be 't' i.e. $(a\cos t,\ b\sin t)$. We know that the co-ordinates of D are $\frac{\pi}{2} + t$ i.e.

$$\left\{a\cos\left(\frac{\pi}{2}+t\right),\ b\sin\left(\frac{\pi}{2}+t\right)\right\}$$

Now the radius of curvature at the point $P(t)$ is

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \qquad ...(1)$$

and the radius of curvature at the point $D\left(\frac{\pi}{2}+t\right)$ is

$$\rho' = \frac{\left\{a^2\sin^2\left(\frac{\pi}{2}+t\right) + b^2\cos^2\left(\frac{\pi}{2}+t\right)\right\}^{3/2}}{ab}$$

$$= \frac{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}{ab} \qquad ...(2)$$

From (1), $a^2\sin^2 t + b^2\cos^2 t = (\rho ab)^{2/3}$

From (2), $a^2\cos^2 t + b^2\sin^2 t = (\rho' ab)^{2/3}$

Adding them,

$$a^2 + b^2 = (\rho ab)^{2/3} + (\rho' ab)^{2/3} = \rho^{2/3}(ab)^{2/3} + \rho'^{2/3}(ab)^{2/3}$$
$$= (ab)^{2/3}(\rho^{2/3} + \rho'^{2/3})$$

Hence the result.

EXAMPLE 8. *Find the radius of curvature at any point of the curve*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta).$$

(P.U. 61A, 72A; B.U. 59S, 62A; R.U. 66A; M.U. 78H, 89H)

SOLUTION : Given that $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$.

$$\therefore \quad \frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta,$$

$$\frac{d^2x}{d\theta^2} = -a\sin\theta,\ \frac{d^2y}{d\theta^2} = a\cos\theta.$$

$\therefore$ From the formula,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{\{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{a^2\cos\theta(1 + \cos\theta) + a^2\sin^2\theta}$$

$$= \frac{a^3\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}^{3/2}}{a^2\{\cos\theta + \cos^2\theta + \sin^2\theta\}} = \frac{a(2 + 2\cos\theta)^{3/2}}{1 + \cos\theta}$$

$$= \frac{a.2^{3/2}(1 + \cos\theta)^{3/2}}{1 + \cos\theta} = 2^{3/2} \cdot a(1 + \cos\theta)^{1/2}$$

$$= 2^{3/2}a\left(2\cos^2\frac{\theta}{2}\right)^{1/2} = 2^{3/2}.\ a2^{1/2}\cos\frac{\theta}{2}$$

$$= 4a\cos\frac{\theta}{2}.$$

Note : At $\theta = 0,\ \rho = 4a$. **(B.U. 55A)**

EXAMPLE 9. *Prove that* $\dfrac{1}{\rho^2} = \left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2$.

(P.U. 55, 65; R.U. 69S; M.U. 64, 75, 78)

SOLUTION : We know that

$$\sin\psi = \frac{dy}{ds} \text{ and } \cos\psi = \frac{dx}{ds}.$$

Differentiating both w.r.t. s, we get respectively

$$\cos\psi \cdot \frac{d\psi}{ds} = \frac{d^2y}{ds^2} \Rightarrow \cos\psi \cdot \frac{1}{\rho} = \frac{d^2y}{ds^2} \qquad ...(1)$$

and $$-\sin\psi \cdot \frac{d\psi}{ds} = \frac{d^2x}{ds^2} \Rightarrow -\sin\psi \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2}. \qquad ...(2)$$

Squaring (1) and (2) and adding, we get

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

EXAMPLE 10. *If ρ be the radius of curvature of a parabola at a point whose distance along the curve from a fixed point is s, prove that*

$$3\rho\frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9 = 0. \qquad \textbf{(P.U. 44H, 56H, 61H; Bhag. 63H, 65H; B.U. 67H; M.U. 69H, 70H; R.U. 73H)}$$

SOLUTION : Let the equation of the parabola be $y^2 = 4ax$, so that

$$y = \sqrt{4a}\sqrt{x} = 2\sqrt{a}\sqrt{x}.$$

$$\therefore \quad y_1 = 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$

and $$y_2 = \sqrt{a}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}}.$$

Hence $$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{\left\{1 + \frac{a}{x}\right\}^{3/2}}{-\frac{1}{2} \cdot \frac{\sqrt{a}}{x^{3/2}}}$$

$$= -\frac{(x+a)^{3/2}}{x^{3/2}} \cdot \frac{2x^{3/2}}{\sqrt{a}} = \frac{2(x+a)^{3/2}}{\sqrt{a}} \qquad ...(1)$$

(in magnitude)

We have, $$\frac{d\rho}{ds} = \frac{d\rho}{dx} \cdot \frac{dx}{ds}. \qquad ...(2)$$

Now from (1), $$\frac{d\rho}{dx} = \frac{2}{\sqrt{a}} \cdot \frac{3}{2}(x+a)^{\frac{3}{2}-1} = \frac{3}{\sqrt{a}}\sqrt{(x+a)}.$$

Also $$\frac{ds}{dx} = \sqrt{1 + y_1^2} = \sqrt{1 + \frac{a}{x}} = \sqrt{\frac{x+a}{x}}.$$

$$\therefore \quad \frac{dx}{ds} = \sqrt{\frac{x}{x+a}}.$$

Hence from (2), we have

$$\frac{d\rho}{ds} = \frac{3}{\sqrt{a}} \cdot \sqrt{x+a}\sqrt{\frac{x}{x+a}} = \frac{3\sqrt{x}}{\sqrt{a}}.$$

Again, $$\frac{d^2\rho}{ds^2} = \frac{d}{ds}\left(\frac{d\rho}{ds}\right) = \frac{d}{ds}\left(\frac{3\sqrt{x}}{\sqrt{a}}\right) = \frac{d}{dx}\left(\frac{3\sqrt{x}}{\sqrt{a}}\right) \cdot \frac{dx}{ds}$$

$$= \frac{3}{\sqrt{a}} \cdot \frac{1}{2\sqrt{x}}\sqrt{\frac{x}{x+a}} = \frac{3}{2} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{x+a}}.$$

Thus $$3\rho\frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9$$

$$= 3 \cdot \frac{2}{\sqrt{a}}(x+a)^{3/2} \cdot \frac{3}{2} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{x+a}} - \frac{9x}{a} - 9$$

$$= \frac{9}{a}(x+a) - \frac{9x}{a} - 9 = \frac{9}{a}(x+a-x-a) = 0.$$

Hence the result.

Example 24

1. Find the radius of curvature at any point (s, ψ) of the following curves:

(i) $x = 4a\sin\psi$. (ii) $s = a\tan\psi$. (iii) $s = c\log\sec\psi$.

(iv) $s = a\log\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$.

2. Find the radius of curvature at the point (x, y) of the following curves :

(i) $y^2 = 4ax$. (ii) $xy = c^2$. (iii) $ay^2 = x^3$. (iv) $y = ae^{x/a}$.

(v) $x^{2/3} + y^{2/3} = a^{2/3}$. **(P.U. 60S)** (vi) $y = a\log\sec\frac{x}{a}$. **(Bhag. 94)**

(vii) $y = c\cos h\frac{x}{c}$. **(P.U. [illegible]U. 59; R.U. 68A; Bhag. 91)**

(viii) $y = 4\sin x - \sin 2x$ at $x = \frac{\pi}{2}$. **(P.U. 57H; M.U. 68H)**

(ix) $x^2y = a^2(x-y)$ at $x = a$. **(P.U. 35S)**

(x) $x^2y = a(x^2+y^2)$ at the point $(-2a, 2a)$ **(B.U. 92H)**

[Here $\frac{dy}{dx} = 0$ at the given point. So use $\frac{dx}{dy}$.]

3. Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

(B.U. 90H)

4. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the radius of curvature at the end of the major axis is equal to the semi-latus rectum of the ellipse. **(R.U. 67A, 72A)**

5. Find the radius of curvature of the curve $y=e^x$ at the point where it crosses the y-axis. **(P.U. 66A; M.U. 70A)**

6. Find the radius of curvature at the point $(-4, 0)$ on the curve $xy^2=16(x+4)$. **(P.U. 57A)**

[Here $y_1=\infty$ at $(-4, 0)$] .

7. Show that the radius of curvature at a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a\sin\theta\cos\theta$. **(P.U. 60S, 66S, 90H; B.U. 91H)**

8. Show that the radius of curvature at a point of the curve $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ is at.

9. Show that the radius of curvature at a point of the given curve $x=a\sin2\theta(1+\cos2\theta)$, $y=a\cos2\theta(1-\cos2\theta)$ is $4a\cos3\theta$ and at $\theta=0$, $\rho=4a$. **(P.U. 55S; B.U. 55A; Bhag. 63S)**

10. Find the radius of curvature for the curve $y=\sqrt{s^2+c^2}$. **(B.U. 63, 65, 69S)**

11. Show that for the curve $s^2=8ay$, $\rho=4a\left(1-\frac{y}{2a}\right)^{1/2}$ **(P.U. 52S; M.U. 64S, 65A, 71A; Bhag. 65A)**

Soln. : We have $s^2=8ay$.

Differentiating w.r.t. y,

$$2s\frac{ds}{dy}=8a \Rightarrow \frac{dy}{ds}=\frac{s}{4a}$$

$$\Rightarrow \quad \frac{s}{4a}=\sin\psi \quad \therefore \quad s=4a\sin\psi.$$

Hence $\frac{ds}{d\psi}=4a\cos\psi$

$$\Rightarrow \quad \rho=4a\sqrt{1-\sin^2\psi}.$$

$$=4a\sqrt{1-\frac{s^2}{16a^2}}=4a\sqrt{1-\frac{8ay}{16a^2}}$$

$$=4a\left(1-\frac{y}{2a}\right)^{1/2}.$$

12. Prove that for the curve $y = c\cosh\frac{x}{c}$ the radius of curvature at any point varies as the square of the ordinate at that point.

(P.U. 58A; B.U. 59A; Bhag. 69S)

13. Prove that the radius of curvature of the conic $ax^2 + by^2 = 1$ varies as the cube of the length of the normal. **(P.U. 57S)**

14. Show that in the parabola $y^2 = 4ax$, the radius of curvature at any point P is twice the part of the normal intercepted between the curve and the directrix.

[**Hint :** The eqn. of the normal at $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$

It will intersect the directrix $x = -a$ at the point $(-a, 3at + at^3)$.

The intercept between the directrix and the curve = distance between $(at^2, 2at)$ and $(-a, 3at + at^3)$].

15. Show that the radius of curvature for the curve $y = a\cosh\frac{x}{c}$ is equal to the normal intercepted between the curve and the x-axis and that it varies as the square of the ordinate. **(P.U. 58A; B.U. 59A)**

16. Show that the radius of curvature at any point of the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is twice the portion of the normal intercepted between the curve and the x-axis.

17. For any curve, prove that $\rho = \left\{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2\right\}^{\frac{1}{2}}$.

[**Hint :** Use $\cos\psi = \frac{dx}{ds}$, $\sin\psi = \frac{dy}{ds}$.]

18. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}.$$

[**Hint :** The parametric equations of parabola are $x = at^2$, $y = 2at$. If 't_1' and 't_2' be the ends of a focal chord, then $t_1 t_2 = -1$.]

ANSWER

1. (i) $4a\cos\psi$ (ii) $a\sec^2\psi$ (iii) $c\tan\psi$ (iv) $a\sec\psi$

2. (i) $2a^{-1/2}(a + x)^{3/2}$ (ii) $\frac{(x^2 + y^2)^3}{2c^2}$ (iii) $(4a + 9x)^{3/2} \cdot \frac{\sqrt{x}}{6a}$

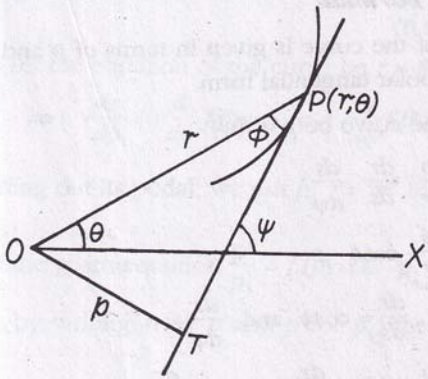
(iv) $\frac{a(1+e^{2x/a})^{3/2}}{e^{x/a}}$ (v) $3(axy)^{1/3}$ (vi) $a\sec\frac{x}{a}$ (vii) $\frac{y^2}{c}$ (viii) $\frac{5}{4}\cdot\sqrt{5}$

(ix) $\pm 2a$. (x) $2a$

3. $\frac{3\sqrt{2}a}{16}$ **5.** $\pm 2\sqrt{2}$ **6.** 2 **10.** $\frac{y^2}{c}$

IV. Pedal Formula

To find the radius of curvature for the pedal curve $p = f(r)$.



Let the pedal equation of the curve be $p = f(r)$.

Let the co-ordinates of any point P on the curve be (r, θ); then $\angle POX = \theta$. From P, a tangent PT has been drawn to the curve which makes an angle ϕ with OP and an angle ψ with OX. Then from the figure,

$$\psi = \theta + \phi$$

$$\therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}. \qquad \ldots(1)$$

From O, $OT = p$ has been drawn perpendicular on PT.

Therefore $p = r\sin\phi$.

Differentiating w.r.t. r, we get

$$\frac{dp}{dr} = r\cos\phi\frac{d\phi}{dr} + \sin\phi = r\cdot\frac{dr}{ds}\cdot\frac{d\phi}{dr} + \frac{rd\theta}{ds};$$

$$\text{since } \cos\phi = \frac{dr}{ds}, \ \sin\phi = \frac{rd\theta}{ds}$$

$$= r\frac{d\phi}{ds} + r\frac{d\theta}{ds} = r\left(\frac{d\phi}{ds} + \frac{d\theta}{ds}\right)$$

$$= r\frac{d\psi}{ds}; \text{ from (1)}$$

$$= r \cdot \frac{1}{\rho}.$$

$$\therefore \qquad \rho = r\frac{dr}{dp}.$$

This is the required formula, when the equation of the curve is given in the pedal form of (p, r).

V. Polar Tangential Formula

When the equation of the curve is given in terms of p and ψ, then the equation is said to be in polar tangential form.

Let the equation of the curve be $p = f(\psi)$.

Then $$\frac{dp}{d\psi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \cdot \frac{ds}{d\psi}$$

$$= \frac{dp}{dr} \cdot \cos\phi \cdot \rho;$$

$$\because \quad \frac{dr}{ds} = \cos\phi \text{ and } \frac{ds}{d\psi} = \rho.$$

$$= \frac{dp}{dr}\cos\phi \cdot r\frac{dr}{dp} \quad \because \rho = r\frac{dr}{dp}$$

$$= r\cos\phi.$$

Now, $$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi; \quad \because p = r\sin\phi$$

$$= r^2.$$

Differentiating both sides w.r.t. p, we get

$$2p + \frac{d}{dp}\left(\frac{dp}{d\psi}\right)^2 = 2r\frac{dr}{dp}$$

$$\Rightarrow \quad 2p + \frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)^2 \cdot \frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

$$\Rightarrow \quad 2p + 2\frac{dp}{d\psi} \cdot \frac{d^2p}{d\psi^2} \cdot \frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

$$\Rightarrow \quad p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp} = \rho$$

$$\therefore \qquad \rho = p + \frac{d^2p}{d\psi^2}.$$

Example 25

1. Find the radius of curvature at any point (p, r) for the following curves whose pedal equations are

(i) $p^2 = ar$ (ii) $pr = a^2$ (iii) $r^3 = a^2p$ (iv) $r^3 = 2ap^2$ **(P.U. 69S)**

(v) $p = \dfrac{r^{m+1}}{a^m}$ **(P.U. 55A, 94H)** (vi) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$

(vii) $p^2(a^2 + b^2 - r^2) = a^2b^2$. **(B.U. 64A)**

2. If ρ and ρ' be the radii of curvature of a curve and of its pedal at corresponding points, show that $\rho'(2r^2 - p\rho) = r^3$.

(B.U. 62H, P.U. 64H; M.U. 73H)

Soln : Let the equation of the curve be $r = f(p)$.

$$\Rightarrow \quad r\frac{dr}{dp} = r\frac{d}{dp}f(p) \quad \Rightarrow \quad \rho = r\frac{d}{dp}f(p).$$

For finding out its pedal, we put $p_1\ r = p^2$ so that $r = \dfrac{p^2}{p_1}$.

$\therefore$ from the given equation $\dfrac{p^2}{p_1} = f(p)$ i.e. $p_1 = \dfrac{p^2}{f(p)}$.

Hence, by writing r for p and p for p_1, the equation of the pedal is

$$p = \frac{r^2}{f(r)}.$$

On differentiation, $\dfrac{dp}{dr} = \dfrac{f(r).2r - r^2\dfrac{d}{dr}f(r)}{\{f(r)\}^2}$

$$\Rightarrow \qquad \rho' = r\frac{dr}{dp} = \frac{r\{f(r)\}^2}{2rf(r) - r^2\dfrac{d}{dr}f(r)}.$$

Hence in terms of original co-ordinates,

$$\rho' = \frac{p\{f(p)\}^2}{2pf(p) - p^2\dfrac{d}{dp}f(p)} = \frac{p.r^2}{2pr - p^2\rho/r}$$

$$= \frac{pr^3}{2pr^2 - p^2\rho} = \frac{r^3}{2r^2 - p\rho} \quad \therefore \quad \rho'(2r^2 - p\rho) = r^3.$$

Answers

1. (i) $\dfrac{2r^{3/2}}{a^{1/2}}$ (ii) $\dfrac{r^2}{a^2}$ (iii) $\dfrac{a}{3r}$ (iv) $\dfrac{2}{3}\sqrt{(2ar)}$ (v) $\dfrac{a^m}{(m+1)r^{m-1}}$ (vi) $\dfrac{a^2b^2}{p^3}$ (vii) $\dfrac{a^2b^2}{p^3}$.

VI. Polar formula

To find the radius of curvature for the polar curve $r = f(\theta)$.

Let the polar equation of the curve be $r = f(\theta)$.

We know from Art. 5.11 that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad ...(1)$$

Differentiating w.r.t. θ, we get

$$\frac{-2}{p^3}\frac{dp}{d\theta} = \frac{-2}{r^3}\frac{dr}{d\theta} + \frac{-4}{r^5}\left(\frac{dr}{d\theta}\right)^3 + \frac{2}{r^4}\left(\frac{dr}{d\theta}\right)\cdot\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \quad \frac{1}{p^3}\frac{dp}{d\theta} = \frac{1}{r^3}\frac{dr}{d\theta} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^3 - \frac{1}{r^4}\frac{dr}{d\theta}\cdot\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \quad \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}\ ;\ \text{multiplying by } \frac{d\theta}{dr}$$

$$= \frac{1}{r^5}\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}.$$

$$\therefore \quad \frac{dp}{dr} = \left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}\cdot\frac{p^3}{r^5}. \qquad ...(2)$$

But from (1), $\dfrac{1}{p^2} = \dfrac{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}{r^4} \Rightarrow p^2 = \dfrac{r^4}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

$$\therefore \quad p = \frac{r^2}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{1/2}}. \qquad ...(3)$$

Now $\because$ $\rho = r\dfrac{dr}{dP}$ $\therefore$ from(2),

$$\rho = r\cdot\frac{r^5}{p^3}\cdot\frac{1}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$$

$$= \frac{r^6\cdot\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^6\cdot\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right\}};$$

putting the value of p from (3)

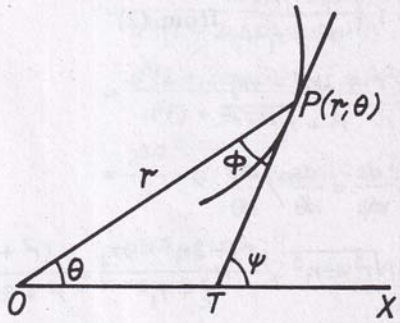
$$= \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Now, if we put $\frac{dr}{d\theta} = r_1$ and $\frac{d^2r}{d\theta^2} = r_2$, then

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}.$$

Alternative Proof

Let the equation of the curve be $r = f(\theta)$.



Let the co-ordinates of any point P on the curve be (r, θ). From P, a tangent PT has been drawn to the curve which makes an angle ϕ with OP and an angle ψ with OX.

We know that Art. 5.9 that

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + r_1^2.$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}. \qquad ..(1)$$

Now, since $\tan\phi = r\frac{d\theta}{dr} = r \Big/ \frac{dr}{d\theta} = \frac{r}{r_1}$,

therefore differentiating w.r.t. θ, we get

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - rr_2}{r_1^2}; \quad r_2 = \frac{d^2r}{d\theta^2}$$

$$= \frac{r_1^2 - rr_2}{r_1^2}$$

$$\Rightarrow \quad (1 + \tan^2\phi)\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\Rightarrow \quad \left(1 + \frac{r^2}{r_1^2}\right)\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r^2 + r_1^2}. \qquad \ldots(2)$$

Again, from the figure, $\psi = \theta + \phi$.

$$\therefore \quad \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$= 1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}; \text{ from (2)}$$

$$= \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}.$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \Big/ \frac{d\psi}{d\theta}.$$

$$= \sqrt{r^2 + r_1^2} \Big/ \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2} = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}.$$

Worked out Examples

EXAMPLE 1. *Find the radius of curvature at any point of the curve* $r = ae^{m\theta}$.

SOLUTION : Given : The equation of the curve is $r = ae^{m\theta}$.

$\therefore$ Differentiating w.r.t. θ, we get

$$\frac{dr}{d\theta} = ae^{m\theta}.m = mr$$

$$\frac{d^2r}{d\theta^2} = m\frac{dr}{d\theta} = m.mr = m^2r.$$

$\therefore$ From the formula, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$$= \frac{(r^2 + m^2r^2)^{3/2}}{r^2 + 2m^2r^2 - m^2r^2} = \frac{\{r^2(1 + m^2)\}^{3/2}}{r^2 + m^2r^2}$$

$$= \frac{r^3(1 + m^2)^{3/2}}{r^2(1 + m^2)} = r\sqrt{1 + m^2}.$$

EXAMPLE 2. *Find the radius of curvature at any point of the curve* $r = a(1 + \cos\theta)$. **(P.U. 45S, 56S; B.U. 56S, 76; R.U. 62A)**

SOLUTION : Given : The equation of the curve is $r = a(1 + \cos\theta)$.

$\therefore$ Differentiating w.r.t. θ, we get

$$\frac{dr}{d\theta} = -a\sin\theta, \quad \frac{d^2r}{d\theta^2} = -a\cos\theta .$$

$\therefore$ From the formula, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$$= \frac{\{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{a^2(1 + \cos\theta)^2 + 2a^2\sin^2\theta + a(1 + \cos\theta).a\cos\theta}$$

$$= \frac{\{a^2(1+ 2\cos\theta + \cos^2\theta + \sin^2\theta)\}^{3/2}}{a^2\{(1 + 2\cos\theta + \cos^2\theta) + 2\sin^2\theta + (\cos\theta + \cos^2\theta)\}}$$

$$= \frac{a^3(2 + 2\cos\theta)^{3/2}}{a^2(3 + 3\cos\theta)} = \frac{a2^{3/2}(1 + \cos\theta)^{3/2}}{3(1 + \cos\theta)}$$

$$= \frac{2^{3/2}}{3} \cdot a(1 + \cos\theta)^{1/2} = \frac{2^{3/2}}{3} \cdot a(2\cos^2\theta)^{1/2}$$

$$= \frac{2^{3/2}.a.2^{1/2}\cos\frac{\theta}{2}}{3} = \frac{4a}{3}\cos\frac{\theta}{2}.$$

Cor. $\rho = \dfrac{4a}{3}\cos\dfrac{\theta}{2} \Rightarrow \rho^2 = \dfrac{16a^2}{9}\cos^2\dfrac{\theta}{2} = \dfrac{16a^2}{9} \cdot \dfrac{1 + \cos\theta}{2}$

$$= \frac{8a^2}{9} \cdot \frac{r}{a} = \frac{4}{9} \cdot 2ar$$

$\therefore$ $$\rho = \frac{2}{3}\sqrt{2ar}.$$ **(B.U. 93H)**

EXAMPLE 3. *Find the radius of curvature at any point of the curve* $r^m = a^m\cos m\theta$. **(P.U. 48S, 50S, 61S, 63A, 67; M.U. 75H, 91; B.U. 53S, 69A; Bhag. 69A, 71A; R.U. 76A)**

SOLUTION : Here the equation of the curve is given in polar form, therefore if we so like, we can find the radius of curvature by applying the formula in the polar form. But it would be easier to find the radius of curvature by transforming the given equation in the pedal form. Therefore, first of all we transform the given equation in the pedal form.

Given : The equation of the curve is $r^m = a^m \cos m\theta$.

Now taking log on both sides,

$$m \cdot \log r = m \log a + \log \cos m\theta .$$

Differentiating w.r.t. θ, we get,

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{-\sin m\theta}{\cos m\theta} m$$

$$\Rightarrow \quad \frac{r d\theta}{dr} = -\cot m\theta \quad \Rightarrow \quad \tan\phi = -\cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2} + m\theta.$$

Again, $p = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta$

$$= r \cdot \frac{r^m}{a^m}, \text{ from the eqn. of the curve}$$

$$= \frac{r^{m+1}}{a^m}.$$

Now differentiating both sides w.r.t. θ, we get

$$\frac{dp}{dr} = \frac{1}{a^m}(m+1)r^m \quad \Rightarrow \quad \frac{dr}{dp} = \frac{a^m}{(m+1)r^m}$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r \cdot \frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}.$$

EXAMPLE 4. *If* $r = a\sec 2\theta$, *prove that* $\rho = -\frac{r^4}{3p^3}$. **(P.U. 61A, 64A, 68S)**

SOLUTION : We shall first of all, reduce the given equation $r = a\sec 2\theta$ in the pedal form.

Taking logarithm of both sides, we have

$$\log r = \log a + \log \sec 2\theta.$$

Then differentiating w.r.t. θ, we get

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\sec 2\theta} \cdot 2\sec 2\theta \tan 2\theta$$

$$\Rightarrow \quad \tan\phi = \frac{1}{2\tan 2\theta}. \qquad \ldots(1)$$

Again, $p = r\sin\phi$

$$= r \cdot \frac{1}{\sqrt{1 + 4\tan^2 2\theta}}; \text{ from (1)}$$

$$= r \cdot \frac{1}{\sqrt{4(\sec^2 2\theta - 1) + 1}} = r \cdot \frac{1}{\sqrt{4\sec^2 2\theta - 3}}$$

$$\Rightarrow \quad p^2 = r^2 \cdot \frac{1}{4\sec^2 2\theta - 3} = r^2 \cdot \frac{1}{4\left(\frac{r^2}{a^2}\right) - 3}$$

$$= r^2 \cdot \frac{a^2}{4r^2 - 3a^2}$$

$$\Rightarrow \quad p^2(4r^2 - 3a^2) = r^2a^2 \qquad \ldots(2)$$

$$\Rightarrow \quad r^2(4p^2 - a^2) = 3a^2p^2. \qquad \ldots(3)$$

Now differentiating (3) w.r.t. p, we get

$$r^2(8p) + (4p^2 - a^2) \cdot 2r\frac{dr}{dp} = 3a^2 \cdot 2p$$

$$\Rightarrow \quad r^2(4p) + (4p^2 - a^2) \cdot \rho = 3a^2p$$

$$\Rightarrow \quad (4p^2 - a^2)\rho = p(3a^2 - 4r^2)$$

$$\Rightarrow \quad \frac{3a^2p^2}{r^2} \cdot \rho = p\left(-\frac{r^2a^2}{p^2}\right); \text{ from (2) \& (3)}$$

$$\therefore \quad \rho = -\frac{r^4}{3p^3}.$$

Example 26

1. Find the radius of curvature at any point (r, θ) for the following curves whose polar equations are

(i) $r = a\theta$ (ii) $r = ae^{\theta\cot\alpha}$ (iii) $r = a\cos\theta$ **(P.U. 64A)**

(iv) $r = a\sin\theta$ (v) $r = a(1 - \cos\theta)$ **(P.U. 56; R.U. 62; Bhag. 69S)**

(vi) $\frac{1}{r^2} = \frac{1}{a^2}\cos\frac{\theta}{2}$ (vii) $r^2 = a^2\cos 2\theta$ **(B.U. 60S, 65A, 75A; R.U. 67H)**

(viii) $r^2\cos 2\theta = a^2$ (ix) $r^n = a^r\sin n\theta$ **(M.U. 69S)**

(x) $r = \frac{a}{1 + \cos\theta}$ (xi) $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{r}{a}$

2. Show that the radii of curvature of the curves $r = a\theta$ and $r\theta = a$ at their common point are in the ratio 3 : 1.

3. If $r = a\sec 2\theta$, prove that $\rho = -\frac{r^4}{3p^3}$. **(P.U. 61A, 64A, 68S)**

4. If ρ_1 and ρ_2 are the radii of curvature at the extremities of any chord of the cardiode $r = a(1 + \cos\theta)$, which passes through the pole, prove that

$$\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2. \qquad \textbf{(B.U. 68H)}$$

[**Hint :** The extremities of any chord through the pole correspond to θ and $\pi + \theta$.

Then as in Ex. 2 worked out,

$$\rho_1 = \frac{4a}{3}\cos\frac{\theta}{2} \text{ and } \rho_2 = \frac{4a}{3}\cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right).]$$

ANSWERS

1. (i) $\dfrac{a(1+\theta^2)^{3/2}}{2+\theta^2}$ (ii) $r\operatorname{cosec}\alpha$ (iii) $\dfrac{a}{2}$ (iv) $\dfrac{a}{2}$ (v) $\dfrac{2}{3}(2ar)^{1/2}$ (vi) $\dfrac{2}{3}(ar)^{1/2}$

(vii) $\dfrac{a^2}{3r}$ (viii) $\dfrac{r^3}{a^2}$ (ix) $\dfrac{a^m}{(m+1)r^{m-1}}$ (x) $\dfrac{(2r)^{3/2}}{a^{1/2}}$ (xi) $\sqrt{r^2 - a^2}$.

6.3. Curvature at the Origin

We know that

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \qquad \ldots(1)$$

If any curve $y = f(x)$ passes through the origin (0, 0), then in order to use the formula (1) we shall have to evaluate y_1 and y_2 at the point (0, 0), i.e. we shall have to put $x = 0,\ y = 0$ in y_1 and y_2.

In this relation, we write

$$(y_1)(0, 0) = p \text{ and } (y_2)(0, 0) = q.$$

Hence from (1), at the origin

$$\rho = \frac{(1 + p^2)^{3/2}}{q} \qquad \ldots(2)$$

As said above, we get the value of p by putting $x = 0,\ y = 0$ in y_1 and the value of q by putting $x = 0,\ y = 0$ in y_2. But we are going to show that we can find out the values of p and q by another method i.e. we can find their values easily by expansion. This is as follows:

I. Method of Expansion by Maclaurin's Theorem

According to the Maclaurin's Theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

$$\Rightarrow \qquad y = 0 + xp + \frac{x^2}{2!}q + \ldots$$

where $p = f'(0)$ and $q = f''(0)$.

Since the curve passes through the origin, $\therefore f(0) = 0$.

$$\Rightarrow \qquad y = px + \frac{qx^2}{2} + \ldots$$

By putting the value of y in the equation of the curve and equating the similar powers of x, we find out the values of p and q. Thus, by putting the values of p and q in the formula (2), we find out the radius of curvature at the origin.

II. Newtonian Method

(i) Cartesian form—If the curve pasess through the origin (0, 0) and the x-axis is the tangent at the origin it means that when $x = 0$, $y = 0$, then $p = 0$.

If $y = f(x)$ can be expanded according to the Maclaurin's Theorem in the neighbourhood of origin, then

$$y = f(0) + px + q\frac{x^2}{2} + \ldots$$

$$= 0 + 0.x + q\frac{x^2}{2} + \ldots$$

$$= q\frac{x^2}{2} + \text{higher powers of } x.$$

Now dividing both sides by x^2, we get

$$\frac{2y}{x^2} = q + \text{powers of } x$$

$$\therefore \quad \underset{x \to 0}{Lt} \frac{2y}{x^2} = q.$$

Hence putting $p = 0$ in the formula $\rho = \dfrac{(1 + p^2)^{3/2}}{q}$

we get $$\rho = \frac{1}{q} = \underset{x \to 0}{Lt} \frac{x^2}{2y}. \qquad \ldots(1)$$

Similarly it can be shown that if the curve passes through the origin, and the y-axis is the tangent at the origin, then

$$\rho \text{ (at the origin)} = \underset{y \to 0}{Lt} \frac{y^2}{2x}. \qquad \ldots(2)$$

(ii) Polar form—If the equation of the curve is given in polar form and the initial line (x-axis) is tangent at the origin.

In this case $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore$ Writing in polar co-ordinates, the previous formula

$$\rho \text{ (at the pole)} = \underset{x \to 0}{Lt} \frac{x^2}{2y}$$

$$= \underset{\theta \to 0}{Lt} \frac{r^2 \cos^2\theta}{2r \sin\theta}$$

because when $x \to 0$, $y \to 0$, then $\theta \to 0$

$$= \underset{\theta \to 0}{Lt} \frac{r}{2} \frac{\cos^2\theta}{\sin\theta}$$

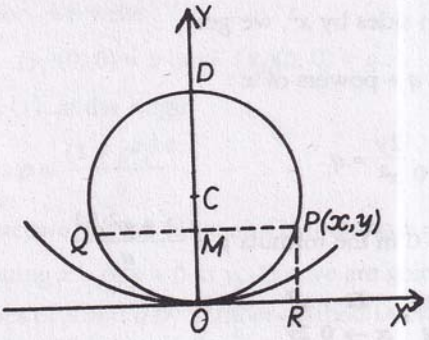
$$= \underset{\epsilon \to 0}{Lt} \left(\frac{r}{2\theta} \cdot \frac{\theta}{\sin\theta} \cdot \cos^2\theta \right)$$

$$= \underset{\theta \to 0}{Lt} \left(\frac{r}{2\theta} \right) \quad \because \underset{\theta \to 0}{Lt} \frac{\theta}{\sin\theta} = 1.$$

6.4. Geometrical Proof of Newtonian Method

Let the x-axis be tangent to the curve at the origin.

Let $P(x, y)$ be any point on the curve which is very near to O. We draw a circle through O and P which touches the x-axis at the point O i.e. the circle passes through two coincident points at O and the third point P.



It is obvious that when P moves along the curve to coincide with O, then this circle becomes the circle of curvature at the point O.

Let OD be the diameter of the circle. From P draw $PQ \perp OD$. Therefore the middle point M of PQ will be on the diameter OD.

Now, from Geometry

$$OM.MD = PM.MQ$$

$$\Rightarrow \quad OM.MD = PM.PM \quad (\because PM = MQ)$$

$$\therefore \quad MD = \frac{PM^2}{OM} = \frac{OR^2}{PR} = \frac{x^2}{y}.$$

Now we coincide the point P with the point O, then $x \to 0$, $y \to 0$ and MD becomes the diameter of the curvature.

Therefore taking limit, $\underset{\substack{x \to 0 \\ y \to 0}}{Lt} MD = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2}{y}$.

or $$2\rho = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2}{y}$$; where ρ = radius of curvature

$$\therefore \quad \rho = \frac{1}{2} \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2}{y}.$$

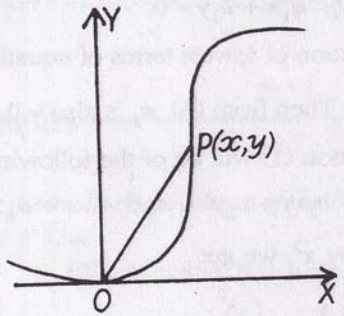
Similarly if y-axis is the tangent at the origin, it can be shown that

$$\rho = \frac{1}{2} \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y^2}{x}.$$

Note : If any curve passes through the origin and we require to find whether x-axis or y-axis is tangent to the curve at the origin we use the following general theorem.

6.5. Tangent at the origin

Statement *—If the equation of a curve which passes through the origin be given in the rational algebraic form, then the equation of the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation of the curve.*



Let the equation of the curve (which passes through the origin) is

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 + \ldots + a_nx^n + \ldots + k_ny^n = 0 \quad \ldots(1)$$

where a_1, b_1, a_2, b_2, c_2 .... are constants.

Let $P(x, y)$ be any point on the curve. Then the equation of the st. line OP will be

$$Y = \frac{y}{x}X, \text{ where } X, Y \text{ are current co-ordinates.}$$

If OP is tangent to the curve at the origin, then its form will be

$$Y = mX \quad \ldots(2)$$

where $m = \underset{P \to 0}{Lt} \frac{y}{x} = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x}$.

Now, dividing the equation (1) by x, we get

$$a_1 + b_1\frac{y}{x} + a_2x + b_2y + c_2y.\frac{y}{x} + \ldots = 0.$$

Taking the limit of both sides when $x \to 0$, $y \to 0$, we find that

$$a_1 + b_1m = 0$$

because all the succeeding terms become zero when $x \to 0$, $y \to 0$.

$$\therefore \quad m = -\frac{a_1}{b_1}. \quad \ldots(3)$$

Case I. We suppose that $b_1 \neq 0$.

Therefore from (2), the equation of the tangent at the origin is

$$Y = -\frac{a_1}{b_1}X \Rightarrow b_1Y = -a_1X \Rightarrow a_1X + b_1Y = 0.$$

i.e. in terms of (x, y); $a_1x + b_1y = 0$. ...(4)

But (4) is the collection of lowest terms of equation (1).

Case II. Let $b_1 = 0$. Then from (3), a_1 is also = 0 ($\because$ m is finite).

Therefore the equation (1) will be of the following form :

$$a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + \ldots + a_nx^n + \ldots + k_ny^n = 0.$$

Now dividing this by x^2, we get

$$a_2 + b_2\frac{y}{x} + c_2\left(\frac{y}{x}\right)^2 + \ldots = 0.$$

Taking the limit of both sides when $x \to 0$, $y \to 0$, we find that

$$a_2 + b_2 m + c_2 m^2 = 0 \qquad \ldots(5)$$

because all the succeeding terms become zero when $x \to 0$, $y \to 0$. But this is a quadratic equation in m. This means that in this case, there will be two tangents at the origin.

Now putting $m = \frac{Y}{X}$ in (5), $a_2 + b_2\frac{Y}{X} + c_2\frac{Y^2}{X^2} = 0$

$$\Rightarrow \qquad a_2X^2 + b_2XY + c_2Y^2 = 0.$$

But when $a_1 = 0$, $b_1 = 0$, then the equation of the lowest degree terms of the equation (1) is $a_2x^2 + b_2xy + c_2y^2 = 0$.

Hence in this case also, the theorem is proved.

Similarly if $a_2 = 0 = b_2 = c_2$, then the truth of the theorem can be proved as before.

EXAMPLE 1. *We consider the parabola* $y^2 = 4ax$.

SOLUTION : Here the term of the lowest degree is $4ax$.

$\therefore \quad 4ax = 0$ i.e. $x = 0$.

is the tangent to the parabola at the origin. That is, y-axis is the tangent to the parabola at the origin.

EXAMPLE 2. *We consider the curve* $x^3 + y^3 = 3axy$.

SOLUTION : Here the term of the lowest degree is $3axy$.

$\therefore$ Putting $3axy = 0$ we get $xy = 0$ i.e. $x = 0$, $y = 0$ i.e. x-axis and y-axis both are tangents to the curve at the origin.

Working Rule—If the x-axis or the y-axis is the tangent to the curve at the origin, then in order to find the radius of curvature at the origin, we should use either Newton's method or the expansion method e.g. Maclaurin's expansion.

Worked out Examples

EXAMPLE 1. *Show that the radii of curvature of the curve* $y^2 = x^2\frac{(a+x)}{a-x}$ *at the origin are* $\pm\sqrt{2}\,a$.

(P.U. 34A, 48A; B.U. 62S; Bhag. 63A; M.U. 67A, 72A)

SOLUTION : Given the equation of the curve $y^2 = \frac{x^2(a+x)}{a-x}$.

$$\Rightarrow \qquad y^2(a-x) = x^2(a+x).$$

According to the rule, putting

$$y = px + q\frac{x^2}{2} + \ldots, \text{ we get}$$

$$(a-x)(px + q\frac{x^2}{2} + \ldots)^2 = x^2(a+x)$$

$$\Rightarrow \qquad (a-x)\left(p^2x^2 + 2p \cdot \frac{q}{2}x^3 + \ldots\right) = x^2(a+x).$$

Now, equating the coeffs. of x^2 and x^3 from both sides, we get

$$ap^2 = a \Rightarrow p^2 = 1 \quad \therefore \quad p = \pm 1$$

and $\quad apq - p^2 = 1 \Rightarrow apq = 2 \quad \therefore \quad q = \dfrac{2}{ap}.$

$\therefore$ If $p = +1$, then $q = \dfrac{2}{a}$; if $p = -1$, then $q = -\dfrac{2}{a}$.

Now, from the formula,

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\pm\frac{2}{a}}$$

$$= \pm\frac{a}{2} \cdot 2^{3/2} = \pm\sqrt{2}\,a.$$

Second Method :

The equation of the given curve can be written as follows:

$$y = \pm\sqrt{\frac{a+x}{a-x}} = \pm x\frac{a+x}{\sqrt{a^2-x^2}}$$

$$= \pm x(a+x)\frac{1}{a\sqrt{1-x^2/a^2}} = \pm x\left(1+\frac{x}{a}\right)\left(1-\frac{x^2}{a^2}\right)^{-1/2}$$

$$= \pm\left(x+\frac{x^2}{a}\right)\left(1+\frac{1}{2}\frac{x^2}{a^2}+\ldots\right); \text{ from Binomial Theorem}$$

$$= \pm\left(x+\frac{1}{a}x^2+\ldots\right)$$

This equation is of the following form :

$$y = px + q\frac{x^2}{2!} + \ldots$$

Therefore on comparison, $p = \pm 1$, and $q = \pm\dfrac{2}{a}$.

Therefore, as before $\rho = \pm\sqrt{2}\,a$.

EXAMPLE 2. *Show that the radii of curvature at the origin for the curve* $x^3 + y^3 = 3axy$ *are each equal to* $\frac{3}{2}a$.

(P.U. 39A, 44S, 51A, 58S, 71A; B.U. 56S, 61A; R.U. 68S, 73A, 80; Bhag. 88, 93)

SOLUTION : Given the equation of the curve $x^3 + y^3 = 3axy$.

Since the curve does not contain any constant term, therefore the curve passes through the origin. Again, we know that in order to find the equation of the tangent at the origin, we need to equate to zero the term of the lowest degree in the equation of the curve.

Here the term of the lowest degree is $3axy$. Therefore the equation of the tangent to the curve at the origin is $3axy = 0$ i.e. $x = 0$, $y = 0$ i.e. both the x-axis and y-axis are tangents to the curve at the origin.

Case I. If the x-axis is the tangent to the curve at the origin, then from Newton's method

$$\rho = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2}{2y}.$$

Here $x^3 + y^3 = 3axy$.

$$\Rightarrow \quad \frac{x^3}{2xy} + \frac{y^3}{2xy} = \frac{3axy}{2xy}$$

$$\Rightarrow \quad \frac{x^2}{2y} + \frac{y^2}{2x} = \frac{3a}{2}.$$

$\therefore$ Taking the limit when $x \to 0$, $y \to 0$,

$$\underset{x,\, y \to 0}{Lt} \left(\frac{x^2}{2y} + \frac{1}{4}xy \cdot \frac{2y}{x^2}\right) = \frac{3a}{2} \quad \Rightarrow \quad \rho + \frac{1}{4} \cdot 0 \cdot 0 \cdot \frac{1}{\rho} = \frac{3a}{2}.$$

$$\therefore \quad \rho = \frac{3a}{2}.$$

Case II. If the y-axis is the tangent to the curve at the origin, then as before, it can be shown that

$$\rho = \underset{x,\, y \to 0}{Lt} \frac{y^2}{2x} = \frac{3a}{2}.$$

Hence the result.

Example 27

1. Show that in the curve $y = x + 3x^2 - x^3$, the radius of curvature at the origin = ·4714 .... and that at the point (1, 3) it is infinite. **(P.U. 32S)**
2. Find the radius of curvature at the origin of the following curves:—
   (i) $a(y^2 - x^2) = x^3$. (ii) $3x^2 + xy + y^2 - 4x = 0$. **(B.U. 68A)**

(iii) $4x^2 - 3xy + y^2 - 3y = 0$. **(B.U. 53A; Bhag. 68A)**

(iv) $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$. **(M.U. 72S)**

(v) $(y - x)(y - 2x) = x^3 + y^3$. **(P.U. 55A; M.U. 66, 86)**

(vii) $r = a \sin n\theta$. **(P.U. 58A, 62A, 68 A; M.U. 71A)**

[At the origin $\rho = \underset{\theta \to 0}{Lt} \left(\frac{r}{2\theta}\right) = \underset{\theta \to 0}{Lt} \frac{a \sin n\theta}{2\theta}$

$= \underset{\theta \to 0}{Lt} \frac{a}{2} \cdot \frac{\sin n\theta}{n\theta} \cdot n = \frac{na}{2}$.]

3. Find the radius of curvature of the curve $y^2 = x^2 \frac{a + x}{a - x}$ at the point $(-a, 0)$. **(B.U. 54A)**

[Hint : Shift the origin to the point $(-a, 0)$.

The equation thus becomes $y^2 = (x - a)^2 \cdot \frac{a + x - a}{a - (x - a)}$

$= (x - a)^2 \cdot \frac{x}{2a - x}$

i.e. $y^2(2a - x) = x(x - a)^2$.

$\therefore$ The equation of the tangent at the origin is $xa^2 = 0$.

Then use $\rho = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y^2}{2x}$.]

ANSWER

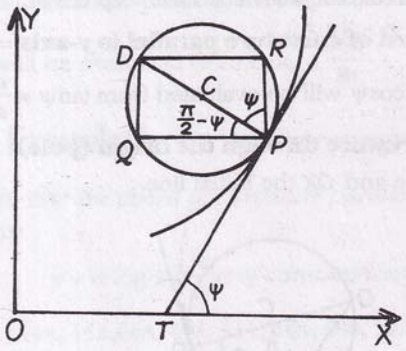
2. (i) $\pm 2a\sqrt{2}$ (ii) 2 (iii) $\frac{3}{8}$ (iv) $\frac{5\sqrt{5}}{2}, -\sqrt{2}$ (v) $\pm \frac{5\sqrt{5}}{18}$ (vi) $\frac{3}{2}$ (vii) $\frac{na}{2}$ **3.** $\frac{a}{4}$.

6.6. Chord of curvature

I. Chord of curvature parallel to the co-ordinate axes in the case of the curve $y = f(x)$.

Let P be any point on the curve $y = f(x)$. Let PQR be the circle of curvature at the point P whose centre is C

In the circle of curvature, from P draw a chord PQ parallel to the x-axis. Then this will be called the chord of curvature parallel to the x-axis.



Similarly, from P we draw a chord PR parallel to y-axis, then this will be called the chord of curvature parallel to y-axis. We seperately find out the lenghts of PQ and PR.

We draw a tangent PT to the curve at the point P which makes an angle ψ with the x-axis.

$\therefore \quad \angle PTX = \psi.$

Join PC and produce it to meet the circle of curvature at the point D.

Then $PD = 2$ (radius of curvature) $= 2\rho$.

Now $\because \quad PQ \parallel OX \quad \therefore \quad \angle QPT = \angle PTX = \psi.$

Again $\because \quad \angle DPT = \dfrac{\pi}{2} \quad \because$ radius is perp. to the tangent..

$\therefore \quad \angle DPQ = \dfrac{\pi}{2} - \psi$

and $\angle DPR = \angle QPR - \angle DPQ = \dfrac{\pi}{2} - \left(\dfrac{\pi}{2} - \psi\right) = \psi.$

Again $\therefore$ angle in a semi-circle is a right angle,

therefore $\quad \angle PQD = \dfrac{\pi}{2}$ and $\angle PRD = \dfrac{\pi}{2}.$

Now (i) from ΔPQD, $\quad PQ = PD \cos DPQ = 2\rho \cos\left(\dfrac{\pi}{2} - \psi\right)$

$$= 2\rho \sin\psi.$$

$\therefore$ **chord of curvature parallel to x-axis $= 2\rho \sin\psi$.**

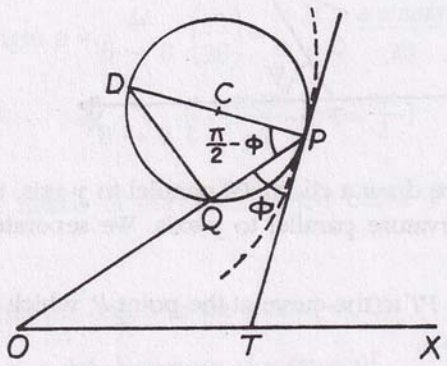
(ii) From ΔPRD, $PR = PD \cos DPR = 2\rho \cos\psi$.

$\therefore$ **chord of curvature parallel to y-axis $= 2\rho\cos\psi$.**

Note : *$\sin\psi$ and $\cos\psi$ will be evaluated from $\tan\psi = \frac{dy}{dx}$.*

II. chord of curvature through the origin (pole).

Let O be the pole and OX the initial line.



Let P be any point on the curve and C be the centre of the circle of curvature at the point P. Join P with the origin O which cuts the circle of curvature at the point Q. Then PQ is the length of the chord of curvature passing through the origin (pole).

We want to find the length of PQ.

Draw a tangent at the point P of the curve which makes an angle ϕ with OP.

Join P with C and produce it to meet the circle of curvature at the point D. Then $PD = 2\rho$.

Join DQ.

Since the angle on the segment of a semi-circle is a right angle, therefore $\angle PQD = 1rt.$ angle.

Again since $DP \perp PT$ $\therefore$ $\angle DPT = \frac{\pi}{2}$

$\therefore$ $\angle DPQ = \frac{\pi}{2} - \phi$.

Now, from ΔDPQ, $PQ = PD \cos DPQ$

$$= 2\rho \cos\left(\frac{\pi}{2} - \phi\right) = 2\rho \sin\phi.$$

$\therefore$ **chord of curvature passing through the origin = $2\rho \sin\phi$.**

Note : $\sin\phi$ will be evaluated from $\tan\phi = \dfrac{rd\theta}{ar}$.

Worked out Examples

EXAMPLE 1. *Prove that the chord of curvature parallel to the axis of y for the curve*

$$y = a \log \sec\frac{x}{a} \text{ is of constant length.}$$

(P.U.30S, 40A, 44A, 45A,46A, 49S, 54A, 64A, 65S, 70A, 86; B.U. 51S, 57A, 62S, 66A, 70A, 85; R.U. 62S, 65A, 69S, 71A, 81; Bhag. 62A, 71A, 92; M.U. 66A, 69A, 71A, 74A, 77, 79, 81; Mithila 76A)

SOLUTION : From the formula, chord of curvature parallel to y-axis

$$= 2\rho \cos\psi.$$

Therefore we shall find out the values of ρ and $\cos\psi$ seperately.

Given the equation of the curve $y = a \log \sec \dfrac{x}{a}$.

Therefore differentiating w.r.t. x

$$\frac{dy}{dx} = a \cdot \frac{1}{\sec\frac{x}{a}} \cdot \sec\frac{x}{a} \cdot \tan\frac{x}{a} \cdot \frac{1}{a} = \tan\frac{x}{a} \qquad \ldots(1)$$

and $$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\frac{x}{a}.$$

$\therefore$ $$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{a\left(1 + \tan^2\frac{x}{a}\right)^{3/2}}{\sec^2\frac{x}{a}} = \frac{a\sec^3\frac{x}{a}}{\sec^2\frac{x}{a}}$$

$$= a\sec\frac{x}{a}.$$

From (1), $\tan\psi = \tan\dfrac{x}{a}$ $\therefore$ $\psi = \dfrac{x}{a}$ $\therefore$ $\cos\psi = \cos\dfrac{x}{a}$.

Hence the chord of curvature parallel to y-axis

$$= 2\rho\cos\psi = 2a\sec\frac{x}{a}\cos\frac{x}{a} = 2a = \text{constant.}$$

EXAMPLE 2. *Prove that the chord of curvature through the pole of the curve*

$$p = f(r) \text{ is } \frac{2f(r)}{f'(r)}.$$

(P.U. 46S, 62A)

SOLUTION : From the formula, the chord of curvature passing through the pole = $2\rho \sin\phi$.

Here $\rho = r\frac{dr}{dp}$ and $p = r\sin\phi$.

$\therefore$ chord of curvature = $2\rho \sin\phi$

$$= 2r\frac{dr}{dp}\cdot\frac{p}{r} = 2p \Big/ \frac{dp}{dr} = \frac{2f(r)}{f'(r)}.$$

Note : This result should be remembered as a formula.

EXAMPLE 3. *Find the chord of curvature through the pole of the curve*

$$r = ae^{\theta \cot\alpha}. \qquad \textbf{(P.U. 49A, 63A)}$$

SOLUTION : Given : The equation of the curve is $r = ae^{\theta \cot\alpha}$.

$\therefore$ $$\frac{dr}{d\theta} = ae^{\theta \cot\alpha}\cdot\cot\alpha = r\cot\alpha$$

and $$\frac{d^2r}{d\theta^2} = \cot\alpha\cdot\frac{dr}{d\theta} = r\cot\alpha\cdot\cot\alpha = r\cot^2\alpha.$$

$\therefore$ $$\rho = \frac{(r^2 + r_1^{\,2})^{3/2}}{r^2 + 2r_1^{\,2} - rr_2}$$

$$= \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha} = \frac{\{r^2(1 + \cot^2\alpha)\}^{3/2}}{r^2 + r^2\cot^2\alpha}$$

$$= \frac{r^3\cdot\operatorname{cosec}^3\alpha}{r^2\cdot\operatorname{cosec}^2\alpha} = r\operatorname{cosec}\alpha.$$

Again, $r\frac{d\theta}{dr} = r/r\cot\alpha = \tan\alpha$..

$\Rightarrow$ $\tan\phi = \tan\alpha$ $\quad\therefore\quad \phi = \alpha$.

$\therefore$ The chord of curvature passing through the pole is

$$= 2\rho\sin\phi = 2r\operatorname{cosec}\alpha\cdot\sin\alpha = 2r.$$

EXAMPLE 4. *Find the chord of curvature through the pole of the curve* $r = a(1 + \cos\theta)$. **(P.U. 47A, 51S, 62S, 87; Bhag. 63; B.U. 57 S, 58A, 63S)**

SOLUTION : Given : The equation of the curve is $r = a(1 + \cos\theta)$.

$\therefore$ $$\frac{dr}{d\theta} = a(-\sin\theta) \quad\text{and}\quad \frac{d^2r}{d\theta^2} = -a\cos\theta.$$

$$\therefore \quad \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2(1+\cos\theta)\cos\theta}$$

$$= \frac{\{a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}^{3/2}}{a^2\{1 + 2\cos\theta + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\}}$$

$$= \frac{\{a^2(2 + 2\cos\theta)\}^{3/2}}{a^2(3 + 3\cos\theta)} = \frac{a^3 \cdot 2^{3/2}(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$= \frac{2^{3/2}a(1+\cos\theta)^{1/2}}{3} = \frac{2^{3/2}a}{3} \cdot \left(2\cos^2\frac{\theta}{2}\right)^{1/2}$$

$$= \frac{2^{3/2}a}{3} \cdot 2^{1/2} \cdot \cos\frac{\theta}{2} = \frac{4a}{3}\cos\frac{\theta}{2}.$$

Again, $$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

$$= \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right).$$

$$\therefore \quad \phi = \frac{\pi}{2} + \frac{\theta}{2}.$$

$\therefore$ The chord of curvature passing through the pole is

$$= 2\rho\sin\phi = 2 \cdot \frac{4a}{3}\cos\frac{\theta}{2} \cdot \sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$= \frac{8a}{3}\cos^2\frac{\theta}{2} = \frac{8a}{3} \cdot \frac{1+\cos\theta}{2} = \frac{4a}{3} \cdot \frac{r}{a} = \frac{4r}{3}.$$

EXAMPLE 5. *Show that the chord of curvature through the pole of the curve* $r^m = a^m \cos m\theta$ *is* $\frac{2r}{m+1}$.

(P.U 35S, 56A, 69A; B.U. 61S, 67A; M.U. 64, 87; R.U. 64A, 69A; Bhag. 66A, 72A)

SOLUTION : Given : The equation of the curve is $r^m = a^m \cos m\theta$.

Now, taking log on both sides, we get

$$m\log r = m\log a + \log\cos m\theta.$$

Again, differentiating both sides w.r.t. θ,

$$m \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin m\theta}{\cos m\theta} \cdot m$$

$$\Rightarrow \quad \frac{rd\theta}{dr} = -\cot m\theta \Rightarrow \tan\phi = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2} + m\theta.$$

Now $$p = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right)$$

$$= r\cos m\theta = r \cdot \frac{r^m}{a^m}, \text{ from the eqn. of the curve}$$

$$= \frac{r^{m+1}}{a^m}.$$

$$\therefore \quad \frac{dp}{dr} = \frac{(m+1)r^m}{a^m}.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r \cdot \frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}.$$

Hence from the formula, the chord of curvature passing through the pole is

$$= 2\rho\sin\phi = 2 \cdot \frac{a^m}{(m+1)r^{m-1}} \cdot \sin\left(\frac{\pi}{2} + m\theta\right)$$

$$= \frac{2a^m}{(m+1)r^{m-1}}\cos m\theta = \frac{2a^m}{(m+1)r^{m-1}} \times \frac{r^m}{a^m}$$

$$= \frac{2r}{m+1}.$$

EXAMPLE 6. *Show that in a parabola, the chord of curvature (i) through the focus (ii) parallel to x-axis are each equal to four times the focal distance of the point* **(B.U. 67; M.U..77H)**

SOLUTION : If we take focus as the pole, then the equation of the parabola will be

$$\frac{2a}{r} = 1 + \cos\theta. \quad (\text{here } l = 2a,\ e = 1)$$

Now taking log on both sides,

$$\log 2a - \log r = \log(1 + \cos\theta).$$

Again, differentiating both sides w.r.t. θ,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\Rightarrow \quad \frac{rd\theta}{dr} = \cot\frac{\theta}{2} \Rightarrow \tan\phi = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\therefore \quad \phi = \frac{\pi}{2} - \frac{\theta}{2}.$$

Now $p = r\sin\phi = r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$.

$$\therefore \quad p^2 = r^2\cos^2\frac{\theta}{2} = \frac{r^2(1+\cos\theta)}{2} = \frac{r^2}{2}\cdot\frac{2a}{r} = ar$$

$$\Rightarrow \quad p = \sqrt{a}\sqrt{r}.$$

$$\therefore \quad \frac{dp}{dr} = \sqrt{a}\cdot\frac{1}{2\sqrt{r}} = \frac{1}{2}\cdot\frac{\sqrt{a}}{\sqrt{r}}.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r\cdot\frac{2\sqrt{r}}{\sqrt{a}} = \frac{2}{\sqrt{a}}r^{3/2}.$$

$\therefore$ The chord of curvature through the pole

$$= 2\rho\sin\phi = 2\cdot\frac{2}{\sqrt{a}}r^{3/2}\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= \frac{4}{\sqrt{a}}\cdot r^{3/2}\cos\frac{\theta}{2}. \qquad \ldots(1)$$

But $1 + \cos\theta = \frac{2a}{r} \Rightarrow 2\cos^2\frac{\theta}{2} = \frac{2a}{r}$

$$\Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{a}{r} \quad \therefore \quad \cos\frac{\theta}{2} = \frac{\sqrt{a}}{\sqrt{r}}.$$

$$\therefore \text{ (i)} = \frac{4}{\sqrt{a}}r^{3/2}\cdot\frac{\sqrt{a}}{\sqrt{r}} = 4r$$

$= 4 \times$ focal distance of the point.

(ii) We have, $\psi = \theta + \phi = \theta + \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} + \frac{\theta}{2}$.

Here $\rho = \frac{2}{\sqrt{a}}r^{3/2}$ (this has been found out before)

and $\sin\psi = \sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$.

$\therefore$ chord of curvature parallel to x-axis

$$= 2\rho \sin\psi = 2 \cdot \frac{2}{\sqrt{a}} r^{3/2} \cdot \cos\frac{\theta}{2}$$

$$= \frac{4}{\sqrt{a}} r^{3/2} \cos\frac{\theta}{2}. \qquad \ldots(2)$$

This becomes exactly of the form (1).

Hence, as before, (2) = $4r$ = 4 × focal distance of the point.

Example 28

1. Show that the chord of curvature through the pole of the curve $r = ae^{m\theta}$ is $2r$.

(P.U. 41S, 47S, 68A, 73A; B.U. 66S; Bhag. 70A; M.U. 73A, 80)

2. Show that the chord of curvature through the pole of the curve $r = a(1 - \cos\theta)$ is $\frac{4}{3}r$. **(B.U. 57S)**

3. In the curve $r^2 = a^2 \cos 2\theta$, show that the chord of curvature through the origin is $\frac{2r}{3}$. **(B.U. 56A; M.U. 65S)**

4. Show that the chord of curvature parallel to the axis of y for the curve $y = c \cosh \frac{x}{c}$ is double the ordinate. **(R.U. 65; B.U. 94H)**

5. If c_x and c_y be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$, prove that

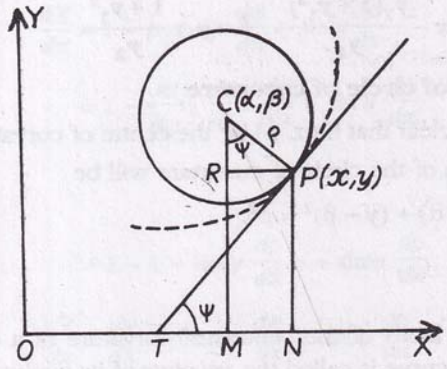
$$\frac{1}{c_x^2} + \frac{1}{c_y^2} = \frac{1}{2ac_x}.$$ **(M.U. 75A)**

6. Show that in a parabola, the chord of curvature through the focus is four times the focal distance of the point. **(B.U. 67A)**

7. Show that (i) the chord of curvature at any point of the cardioid $r = a(1 + \cos\theta)$ is $\frac{2}{3}\sqrt{(2ar)}$ **(Bhag. 63S; P.U. 76A)**

(ii) prove that ρ^2/r is constant. **(P.U. 56S; R.U. 64S, 66S; B.U. 56S, 76A)**

6.7. Centre of curvature



Let $P(x, y)$ be any point on the curve and let C be the centre of the circle of curvature at the point P. Then we need to find the co-ordinates of C.

Let the co-ordinates of C be (α, β). Draw a tangent PT at the point P to the curve which makes an angle ψ with the x-axis.

From C draw CM; and from P draw PN perpendiculars on OX and from P draw PR perp. on CM. Join CP, then

$$CP = \rho \text{ and } \angle RCP = \psi.$$

Now $\alpha = OM = ON - MN = ON - RP = x - CP \sin\psi = x - \rho \sin\psi$. ...(1)

and $\beta = CM = CR + RM = CP \cos\psi + PN = \rho \cos\psi + y$. ...(2)

But we know that

$$\tan\psi = \frac{dy}{dx} = y_1.$$

$$\therefore \quad \sin\psi = \frac{y_1}{\sqrt{1 + y_1^2}} \text{ and } \cos\psi = \frac{1}{\sqrt{1 + y_1^2}}$$

and $$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}.$$

$$\therefore \text{ from (i), } \alpha = x - \frac{(1 + y_1^2)^{3/2}}{y_2} \times \frac{y_1}{\sqrt{1 + y_1^2}} = x - \frac{y_1(1 + y_1^2)}{y_2}.$$

$$\text{and from (ii), } \beta = y + \frac{(1 + y_1^2)^{3/2}}{y_2} \times \frac{1}{\sqrt{1 + y_1^2}} = y + \frac{1 + y_1^2}{y_2}.$$

Thus the co-ordinates of the centre of curvature are

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \beta = y + \frac{1+y_1^2}{y_2}.$$

Note : Equation of circle of curvature

From above, it is clear that if (α, β) be the centre of curvature and radius $= \rho$, then the equation of the circle of curvature will be

$$(x - \alpha) + (y - \beta)^2 = \rho^2.$$

6.8. Evolute

Definition—The locus of the centres of curvature of a curve is called *evolute* and the given curve is called the *involute* of its evolute.

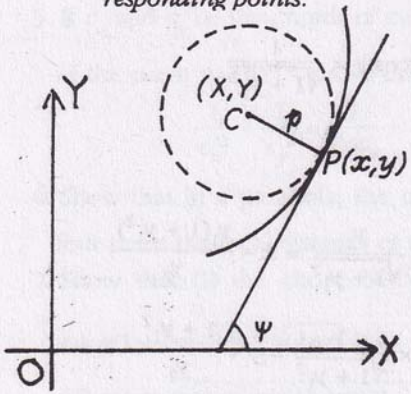
If we need to find the equation of evolute of any curve, then we shall need to eliminate x and y from the co-ordinates α, β of the centre of curvature and the equation of the curve.

Second Definition—It is evident (from the fig. of Art. 6.7) that the centre of curvature is the point of intersection of two consecutive normals to the curve. Since the locus of the centres of curvature is called an evolute, hence we may also *define that an evolute is the envelope of the normals to the given curve.* [See chapter 10]

6.9. Properties of the evolute

(i) *The normal at any point of the curve is the tangent to the evolute at the centre of curvature for that point.*

(ii) *The difference between the radii of curvature at any two points on a curve is equal to the length of the arc of the evolute between the corresponding points.*



Proof. Let $C(X, Y)$ be the centre of curvature for any point $P(x, y)$ on the given curve so that we have

$$X = x - \rho \sin\psi \qquad ...(1)$$

$$Y = y + \rho \cos\psi \qquad ...(2)$$

where ρ is the radius of curvature and ψ is the angle which the tangent at P makes with the x-axis.

Differentiating (1) and (2) w.r.t. x, we get

$$\frac{dX}{dx} = 1 - \rho\cos\psi\frac{d\psi}{dx} - \sin\psi\frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx} - \sin\psi\frac{d\rho}{dx}; \quad \because \rho = \frac{ds}{d\psi} \text{ and } \cos\psi = \frac{dx}{ds}$$

$$= 1 - 1 - \sin\psi\frac{d\rho}{dx} = -\sin\psi\frac{d\rho}{dx} \quad ...(3)$$

And $$\frac{dY}{dx} = \frac{dy}{dx} - \rho\sin\psi\frac{d\psi}{dx} + \cos\psi\frac{d\rho}{dx}$$

$$= \frac{dy}{dx} - \frac{dx}{d\psi}\cdot\frac{dy}{ds}\cdot\frac{d\psi}{dx} + \cos\psi\frac{d\rho}{dx}; \quad \because \sin\psi = \frac{dy}{ds}.$$

$$= \cos\psi\frac{d\rho}{dx} \quad ....(4)$$

Dividing (4) by (3), we get

$$\frac{dY}{dX} = \frac{\cos\psi}{-\sin\psi} = -\cot\psi = -\frac{1}{\frac{dy}{dx}}.$$

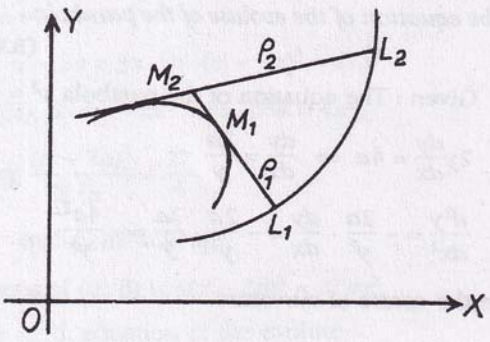
This shows that the tangent to the evolute at $C(X, Y)$ is normal to the curve at $P(x, y)$.

This proves (i)

Again squaring (3) and (4) and then adding, we get

$$\left(\frac{dX}{dx}\right)^2 + \left(\frac{dY}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2 \quad ...(5)$$

Let S denote the length of the arc of the evolute measured from a fixed point on it.



Then we have

$$\left(\frac{dS}{dx}\right)^2 = \left(\frac{dX}{dx}\right)^2 + \left(\frac{dY}{dx}\right)^2 \qquad ...(6)$$

Hence from (5) and (6),

$$\left(\frac{dS}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2 \Rightarrow \frac{dS}{dx} = \frac{d\rho}{dx}$$

$\Rightarrow$ $S = \rho + \lambda$ where λ is any constant.

Let M_1 and M_2 be the two points on the evolute corresponding to two points L_1 and L_2 on the original curve. Let ρ_1 and ρ_2 be the values of ρ for L_1 and L_2 and let S_1 and S_2 be the values of S for M_1 and M_2.

Then $S_1 = \rho_1 + \lambda$ and $S_2 = \rho_2 + \lambda$.

On subtraction, $S_2 - S_1 = \rho_2 - \rho_1$

i.e. the arc M_1M_2 = difference between the radii of curvature at L_1, L_2.

Hence the result.

Cor. Since $(dS)^2 = (d\rho)^2$, therefore

$$\left(\frac{dS}{d\psi}\right)^2 = \left(\frac{d\rho}{d\psi}\right)^2 \Rightarrow \frac{dS}{d\psi} = \frac{d\rho}{d\psi}$$

$\Rightarrow$ $\rho' = \dfrac{d\rho}{d\psi}$ **where ρ' is the radius of curvature of the evolute.**

Below we shall derive the equation of the evolute corresponding to some well-known curves.

EXAMPLE 1. *Find the co-ordinates of the centre of curvature for any point (x, y) on the parabola $y^2 = 4ax$.*

Also, find the equation of the evolute of the parabola.

(B.U. 68H; R.U. 69H)

SOLUTION : Given : The equation of the parabola $y^2 = 4ax$.

$$\therefore \quad 2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

and $$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\cdot\frac{dy}{dx} = -\frac{2a}{y^2}\cdot\frac{2a}{y} = -\frac{4a^2}{y^3}.$$

Let (α, β) be the centre of curvature.

Then, from the formula,

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{-\frac{4a^2}{y^3}}$$

$$= x + \frac{\frac{2a}{y}\cdot\frac{y^2+4a^2}{y^2}}{\frac{4a^2}{y^3}} = x + \frac{2a(y^2+4a^2)}{4a^2}$$

$$= x + \frac{(4ax+4a^2)}{2a} = x + \frac{4a(x+a)}{2a}$$

$$= x + 2(x+a) = 3x + 2a. \qquad \ldots(1)$$

and $$\beta = y + \frac{1+y_1^2}{y_2} = y + \frac{1+\frac{4a^2}{y^2}}{-\frac{4a^2}{y^3}}$$

$$= y - \frac{y^2+4a^2}{y^2}\frac{a^2}{4a^2} \times \frac{y^3}{4a^2} = y - \frac{y(4ax+4a^2)}{4a^2}$$

$$= y - \frac{y\cdot 4a(x+a)}{4a^2} = y - \frac{y(x+a)}{a}$$

$$= y\left\{1 - \frac{x+a}{a}\right\} = y\left(-\frac{x}{a}\right) = -\frac{1}{a}x\cdot\sqrt{4ax}$$

$$= -\frac{2}{\sqrt{a}}x^{3/2}. \qquad \ldots(2)$$

Thus we have found out the co-ordinates (α, β) of the centre of curvature. Now we shall have to eliminate x from (1) and (2) in order to find the equation of the evolute.

From (1), $\alpha - 2a = 3x \Rightarrow (\alpha - 2a)^3 = 27x^3$

and from (2), $\beta\sqrt{a} = -2x^{3/2} \Rightarrow \beta^2 a = 4x^3$.

$\therefore$ dividing $\dfrac{(\alpha-2a)^3}{\beta^2 a} = \dfrac{27}{4}$

$\Rightarrow \quad 4(\alpha - 2a)^3 = 27a\beta^2.$

$\therefore$ The locus of (α, β) is $4(x-2a)^3 = 27ay^2$.

This is the reqd. equation of the evolute.

Cor. Evoluate of the parabola $y^2 = 12x$, by putting $a = 3$ is $2(x - 6)^3 = 81y^2$. **(Bhag. 92H)**

EXAMPLE 2. *Find the co-ordinates of the centre of curvature for any point* (x, y) *on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Hence find the equation of the evolute of the ellipse.*

SOLUTION : We take any point $(a \cos\theta, b \sin\theta)$ on the ellipse i.e. (parametric equation of the ellipse)

$$x = a\cos\theta, \quad y = b\sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta.$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$$

$$= -\frac{b}{a}\frac{d}{d\theta}(\cot\theta) \cdot \frac{d\theta}{dx} = \frac{b}{a}\operatorname{cosec}^2\theta \Big/ \frac{dx}{d\theta}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{1}{-a\sin\theta} = -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

Let (α, β) be the co-ordinates of the centre of curvature.

Therefore

$$\alpha = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta} \cdot \left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - \frac{b}{a}\cot\theta \times \frac{a^2}{b\operatorname{cosec}^3\theta} \times \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}$$

$$= a\cos\theta - \frac{1}{a} \cdot \frac{\cos\theta \times \sin^3\theta}{\sin\theta} \times \frac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin^2\theta}$$

$$= a\cos\theta - \frac{1}{a}\cos\theta(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= \cos\theta\left\{a - \frac{1}{a}(a^2\sin^2\theta + b^2\cos^2\theta)\right\}$$

$$= \cos\theta \cdot \frac{1}{a}\{a^2 - a^2\sin^2\theta - b^2\cos^2\theta\}$$

$$= \cos\theta \cdot \frac{1}{a}(a^2\cos^2\theta - b^2\cos^2\theta)$$

$$= \frac{1}{a}\cos\theta(a^2 - b^2)\cos^2\theta = \frac{a^2 - b^2}{a}\cos^3\theta \qquad ...(1)$$

and $\beta = y + \frac{1}{y_2}(1 + y_1^2)$

$$= b\sin\theta + \frac{1}{-\frac{b}{a^2}\text{cosec}^3\theta}\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta \times \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}$$

$$= b\sin\theta - \frac{1}{b}\sin\theta(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= \sin\theta\left\{b - \frac{1}{b}(a^2\sin^2\theta + b^2\cos^2\theta)\right\}$$

$$= \sin\theta \cdot \frac{1}{b}\{b^2 - a^2\sin^2\theta - b^2\cos^2\theta\}$$

$$= \frac{1}{b}\sin\theta(b^2\sin^2\theta - a^2\sin^2\theta)$$

$$= \frac{b^2 - a^2}{b}\sin^3\theta = -\frac{a^2 - b^2}{b}\sin^3\theta. \qquad ...(2)$$

Thus we have found out the co-ordinates (α, β) of the centre of curvature. Now we shall have to eliminate θ from (1) and (2) in order to find the equation of the evolute.

From (1), $\cos^3\theta = \frac{a\alpha}{a^2 - b^2} \Rightarrow \cos\theta = \left(\frac{a\alpha}{a^2 - b^2}\right)^{1/3}$

From (2), $\sin^3\theta = -\frac{b\beta}{a^2 - b^2} \Rightarrow \sin\theta = -\left(\frac{b\beta}{a^2 - b^2}\right)^{1/3}$

Squaring and adding

$$\cos^2\theta + \sin^2\theta = \left(\frac{a\alpha}{a^2 - b^2}\right)^{2/3} + \left(\frac{b\beta}{a^2 - b^2}\right)^{2/3}$$

$$\Rightarrow \quad \frac{(a\alpha)^{2/3}}{(a^2 - b^2)^{2/3}} + \frac{(b\beta)^{2/3}}{(a^2 - b^2)^{2/3}} = 1$$

$$\Rightarrow \qquad (a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}.$$

$\therefore$ locus of (α, β) is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

This is the reqd. equation of the evolute.

EXAMPLE 3. *Find the co-ordinates of the centre of curvature at any point of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. *Hence find the evolute.* **(M.U. 93H)**

SOLUTION : Let the co-ordinates of any point on the curve be

$$x = a\cos^3\theta \text{ and } y = a\sin^3\theta.$$

Then $$\frac{dy}{d\theta} = a \cdot 3\sin^2\theta\cos\theta = 3a\sin^2\theta\cos\theta;$$

and $$\frac{dx}{d\theta} = a \cdot 3\cos^2\theta(-\sin\theta) = -3a\sin\theta\cos^2\theta \qquad ...(1)$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\sin\theta\cos^2\theta} = -\tan\theta$$

and $$\frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{-\sec^2\theta}{-3a\sin\theta\cos^2\theta} = \frac{1}{3a\sin\theta\cos^4\theta}.$$

Let α, β be the centre of curvature of the evolute. Then

$$\alpha = x - \frac{y_1(1 + y_1^2)}{y_2} \text{ and } \beta = y + \frac{1 + y_1^2}{y_2}.$$

Hence $\alpha = a\cos^3\theta + \tan\theta\sec^2\theta.3a\sin\theta\cos^4\theta$

$= a\cos^3\theta + 3a\sin^2\theta.\cos\theta$

and $\beta = a\sin^3\theta + \sec^2\theta.3a\sin\theta\cos^4\theta$

$= a\sin^3\theta + 3a\sin\theta.\cos^2\theta.$

Adding and subtracting, we get

$$\alpha + \beta = a(\sin\theta + \cos\theta)^3$$

and $$\alpha - \beta = a(\cos\theta - \sin\theta)^3.$$

This $\Rightarrow$ $(\alpha + \beta)^{2/3} = a^{2/3}(\sin\theta + \cos\theta)^2$

and $(\alpha - \beta)^{2/3} = a^{2/3}(\cos\theta - \sin\theta)^2.$

$\therefore$ $(\alpha + \beta)^{2/3} + (\alpha - \beta)^{2/3} = a^{2/3} \times 2$

Hence the locus of (α, β) is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}.$$

EXAMPLE 4. *For the curve* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, *prove that* $\alpha = a(\theta + \sin\theta)$, $\beta = -a(1 - \cos\theta)$ *where* (α, β) *are the co-ordinates of the centre of curvature.*

SOLUTION : From the equation of the given curve, we have

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{1}{2}\frac{d\theta}{dx} = -\frac{1}{2}\frac{\operatorname{cosec}^2\frac{\theta}{2}}{a(1 - \cos\theta)}$$

$$= -\frac{1}{2a} \cdot \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$

$$\text{Now,} \quad \frac{1 + y_1^2}{y_2} = \frac{1 + \cot^2\frac{\theta}{2}}{-\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}} = -4a \cdot \frac{\operatorname{cosec}^2\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}}.$$

$$= -4a\sin^2\frac{\theta}{2}.$$

$\therefore$ From the formula,

$$\alpha = x - \frac{y_1(1 + y_1^2)}{y_2} = a(\theta - \sin\theta) - y_1(-4a\sin^2\frac{\theta}{2})$$

$$= a(\theta - \sin\theta) + \cot\frac{\theta}{2} \cdot 4a\sin^2\frac{\theta}{2}$$

$$= a\theta - a\sin\theta + 4a\cos\frac{\theta}{2}\sin\frac{\theta}{2} = a\theta - a\sin\theta + 2a\sin\theta$$

$$= a\theta + a\sin\theta = a(\theta + \sin\theta);$$

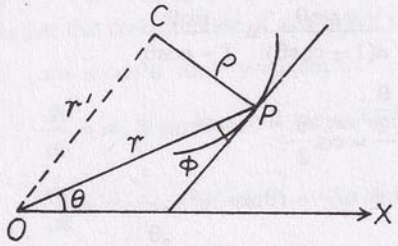
$$\beta = y + \frac{1 + y_1^2}{y_2} = a(1 - \cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a(1 - \cos\theta) - 2a(1 - \cos\theta) = -a(1 - \cos\theta).$$

Note : Thus the parametric equation of the evolute is $x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$.

EXAMPLE 5. *Prove that the evolute of an equiangular spiral is an equiangular spiral of the same angle.*

SOLUTION :



Let C be the centre of the evolute such that $OC = r'$, and let $P(r, \theta)$ be a point on the equiangular spiral $r = ae^{\theta \cot\alpha}$ Let ρ be the radius of curvature of the equi-angular spiral.

We know that in the case of equiangular spiral $r = ae^{\theta \cot\alpha}$, $\phi = \alpha$ and $\rho = \dfrac{r}{\sin\alpha}$.

Now from the ΔOPC,

$$r'^2 = r^2 + \rho^2 - 2r\rho\cos(OPC)$$

$$= r^2 + r^2\operatorname{cosec}^2\alpha - 2r.r\operatorname{cosec}\alpha.\cos(90 - \phi)$$

$$= r^2 + r^2\operatorname{cosec}^2\alpha - 2r^2\operatorname{cosec}\alpha.\sin\alpha; \quad \because \phi = \alpha$$

$$= r^2 + r^2\operatorname{cosec}^2\alpha - 2r^2 = r^2(\operatorname{cosec}^2\alpha - 1)$$

$$= r^2\cot^2\alpha$$

$$\therefore \quad r' = r\cot\alpha.$$

For the vectorial angle of C, we have, again from the ΔOPC,

$$\frac{r'}{\sin(OPC)} = \frac{\rho}{\sin(POC)} \Rightarrow \frac{r\cot\alpha}{\sin(90° - \alpha)} = \frac{r\operatorname{cosec}\alpha}{\sin(POC)}$$

$$\Rightarrow \quad \sin(POC) = \frac{\operatorname{cosec}\alpha.\cos\alpha}{\cot\alpha} = 1 \Rightarrow \angle POC = \frac{\pi}{2}$$

$$\therefore \quad \angle COX = \frac{\pi}{2} + \theta.$$

Thus if the polar co-ordinates of C be (r', θ'), then $r' = r\cot\alpha$ and $\theta' = \frac{\pi}{2} + \theta$.

Hence $r' = ae^{\theta \cot\alpha} \cot\alpha = a\cot\alpha \cdot e^{\left(\theta' - \frac{\pi}{2}\right)\cot\alpha}$

$= a\cot\alpha . e^{\theta' \cot\alpha} . e^{(-\pi/2)\cot\alpha}$

$= \{a \cot\alpha . e^{(-\pi/2)\cot\alpha}\} e^{\theta'\cot\alpha}$

$= be^{\theta'\cot\alpha}$; where $b = \dfrac{a\cot\alpha}{e^{(\pi/2)\cot\alpha}}$

That is, the locus of C is $be^{\theta \cot\alpha}$ which is the equation of the required evolute.

Hence the result.

EXAMPLE 6. *If ρ_1 and ρ_2 be the radii of curvature at the corresponding points of a cycloid and its evolute, prove that $\rho_1^2 + \rho_2^2$ = constant.*

(B.U. 61H, 66H; M.U. 80H)

SOLUTION : Let the equation of the cycloid be

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta) \text{ where } \theta \text{ is the parameter.}$$

Then we have,

$$\frac{dy}{d\theta} = a\sin\theta \text{ and } \frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}. \qquad \ldots(1)$$

Hence $$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{1}{2}\frac{d\theta}{dx} = -\frac{1}{2}\frac{\operatorname{cosec}^2\frac{\theta}{2}}{a(1 - \cos\theta)}$$

$$= -\frac{1}{2a} \cdot \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$

Therefore $\rho_1 = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{(1 + \cot^2\frac{\theta}{2})^{3/2}}{-\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}}$

$$= -4a \cdot \frac{\operatorname{cosec}^3(\theta/2)}{\operatorname{cosec}^4(\theta/2)} = -4a \sin\frac{\theta}{2}.$$

$\therefore$ In magnitude, $\rho_1 = 4a \sin\dfrac{\theta}{2}$.

Now, $\rho_2 = \dfrac{d\rho_1}{d\psi} = \dfrac{d\rho_1}{d\theta} \cdot \dfrac{d\theta}{d\psi} = 4a \cdot \cos\dfrac{\theta}{2} \cdot \dfrac{1}{2}\dfrac{d\theta}{d\psi}$

$$= 2a \cos\frac{\theta}{2}\frac{d\theta}{d\psi} \qquad \ldots(2)$$

But from (1), $\tan\psi = \cot\dfrac{\theta}{2}$ $\therefore$ $\sec^2\psi \dfrac{d\psi}{d\theta} = -\operatorname{cosec}^2\dfrac{\theta}{2} \cdot \dfrac{1}{2}$

$\Rightarrow$ $(1 + \tan^2\psi)\dfrac{d\psi}{d\theta} = -\dfrac{1}{2}\operatorname{cosec}^2\dfrac{\theta}{2}$

$\Rightarrow$ $\left(1 + \cot^2\dfrac{\theta}{2}\right)\dfrac{d\psi}{d\theta} = -\dfrac{1}{2}\operatorname{cosec}^2\dfrac{\theta}{2}$

$\Rightarrow$ $\operatorname{cosec}^2\dfrac{\theta}{2}\dfrac{d\psi}{d\theta} = -\dfrac{1}{2}\operatorname{cosec}^2\dfrac{\theta}{2} \Rightarrow \dfrac{d\psi}{d\theta} = -\dfrac{1}{2}$ $\therefore$ $\dfrac{d\theta}{d\psi} = -2.$

Hence from (2), $\rho_2 = 2a \cos\dfrac{\theta}{2}(-2) = -4a \cos\dfrac{\theta}{2}$

$$= 4a \cos\frac{\theta}{2} \text{ (in magnitude)}$$

Therefore $\rho_1^2 + \rho_2^2 = 16a^2\left(\sin^2\dfrac{\theta}{2} + \cos^2\dfrac{\theta}{2}\right) = 16a^2 = \text{a const.}$

Example 29

1. Prove that the co-ordinates of the centre of curvature at any point (x, y) of any curve can be expressed in the form

$$\bar{x} = x - \frac{dy}{d\psi} \text{ and } \bar{y} = y + \frac{dx}{d\psi}.$$

2. If (α, β) be the co-ordinates of the centre of curvature of the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at (x, y) then $\alpha + \beta = 3(x + y)$.

[**Hint :** $\alpha = x + \sqrt{y} \cdot \dfrac{2(x + y)}{\sqrt{a}}$ and $\beta = y + \dfrac{2\sqrt{x}(x + y)}{\sqrt{a}}$.]

3. Show that $\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$ is the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$

4. Show that the circle of curvature at the origin of the parabola

$$y=mx+\frac{x^2}{a} \text{ is } x^2+y^2=a(1+m^2)(y-mx).$$

5. Prove that for the curve $x=at^2$, $y=2at$, the co-ordinates of the centre of curvature are $\{a(2+3t^2), -2at^3\}$ and hence deduce the equation of its evolute. **(B.U. 53H)**

6. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

7. Find the co-ordinates of the centre of curvature of the rectangular hyperbola $xy=a^2$ and deduce the equation of its evolute.

8. Prove that the evolute of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ is $x=a(\theta+\sin\theta)$, $y=-a(1-\cos\theta)$.

9. Prove that the centre of curvature of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ lies on an equal cycloid.

10. If ρ and ρ' be the radii of curvature at the corresponding points of a curve and its evolute and p, q, r are the first, second and third differential coeffs. of y w.r.t. x, prove that

$$\frac{\rho'}{\rho}=\frac{3pq^2-r(1+p^2)}{q^2}$$ **(P.U. 49H; R.U. 66H)**

[Hints : $\rho'=\frac{d\rho}{d\psi}=\frac{d\rho}{dx}\cdot\frac{dx}{d\psi}=\frac{d\rho}{dx}\cdot\frac{dx}{ds}\cdot\frac{ds}{d\psi}$

$$=\rho\cdot\frac{d\rho}{dx}\cos\psi.$$

Use $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}$.**]**

11. Show that the radius of curvature of the evolute of the curve $r^n=a^n\cos n\theta$ corresponding to (r, θ) is $\frac{n-1}{(n+1)^2}r\sec n\theta\tan n\theta$.

(P.U. 55H; R.U. 64H)

[Hints : Let ρ be the radius of curvature of the given curve. Then

$$\rho=\frac{a^n}{(n+1)r^{n-1}}.$$

$$\therefore \quad \frac{d\rho}{d\psi} = \frac{a^n}{n+1}(1-n)r^{-n}\frac{dr}{d\psi} = \frac{1-n}{n+1}\sec n\theta\frac{dr}{d\psi} \qquad ...(1)$$

But $\phi = \frac{\pi}{2} + n\theta \quad \therefore \quad \psi = \theta + \phi = (n+1)\theta + \frac{\pi}{2}$.

From (1), $\frac{dr}{d\psi} = \frac{dr}{d\theta}\cdot\frac{d\theta}{d\psi} = -r\tan n\theta \cdot \frac{1}{n+1}$

$$\therefore \quad \rho' = \frac{d\rho}{d\psi} = \frac{n-1}{n+1}\sec n\theta\frac{r\cdot\tan n\theta}{n+1}.]$$

12. Prove that in any curve, the radius of curvature of the evolute at a point corresponding to the point (p, r) on the original curve is $\sqrt{r^2 - p^2}\cdot\frac{d}{dp}\left(r\frac{dr}{dp}\right)$. **(P.U. 53H)**

[**Hints :** Here $\rho' = \frac{d\rho}{d\psi} = \frac{d}{d\psi}\left(r\frac{dr}{dp}\right) = \frac{d}{dp}\left(r\frac{dr}{dp}\right)\cdot\frac{dp}{d\psi}$

$$= \frac{d}{dp}\left(r\frac{dr}{dp}\right)\cdot\frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi}$$

$$= \frac{d}{dp}\left(r\frac{dr}{dp}\right)\cdot\frac{dp}{dr}\cdot\cos\phi\,.\,\rho$$

$$= \frac{d}{dp}\left(r\frac{dr}{dp}\right)\cdot\frac{dp}{dr}\cdot\cos\phi\cdot r\frac{dr}{dp}$$

$$= \frac{d}{dp}\left(r\frac{dr}{dp}\right)\cdot r\cos\phi = \frac{d}{dp}\left(r\frac{dr}{dp}\right)\sqrt{r^2 - p^2};$$

since $p = r\sin\phi$.]

13. Prove that the curvature at any point of the pedal of an ellipse of semi-axes a, b with regard to the centre is equal to $\frac{3}{r} - \frac{a^2 + b^2}{r^3}$, where r is the radius vector of the corresponding point of the ellipse. **(B.U. 55H)**

Soln. The pedal equation of the ellipse w.r.t. the centre is

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2} \quad \text{i.e.} \quad \frac{a^2b^2}{p^2} = a^2 + b^2 - r^2 \qquad ...(1)$$

For finding its (first) positive pedal, we write $p_1 r = p^2$ i.e. $r = \frac{p^2}{p_1}$ so that (1) reduces to $\frac{a^2b^2}{p^2} = a^2 + b^2 - \frac{p^4}{{p_1}^2}$.

Therefore by writing r for p and p for p_1, the pedal of (1) is

$$\frac{a^2b^2}{r^2} = a^2 + b^2 - \frac{r^4}{p^2}. \qquad ...(\text{illegible})$$

Let ρ' be the radius of curvature of (2).

Differentiating (2) w.r.t. p, we get

$$a^2b^2\left(\frac{-2}{r^3}\frac{dr}{dp}\right) = -\left\{r^4\left(\frac{-2}{p^3}\right) + \frac{1}{p^2}\cdot 4r^3\frac{dr}{dp}\right\}$$

$$\Rightarrow \quad -a^2b^2\cdot\frac{1}{r^4}\cdot\left(r\frac{dr}{dp}\right) = \frac{r^4}{p^3} - \frac{2r^2}{p^2}\cdot\left(r\frac{dr}{dp}\right)$$

$$\Rightarrow \quad -a^2b^2\cdot\frac{1}{r^4}\cdot\rho' = \frac{r^4}{p^3} - \frac{2r^2}{p^2}(\rho')$$

$$\Rightarrow \quad \rho'\left\{\frac{2r^2}{p^2} - \frac{a^2b^2}{r^4}\right\} = \frac{r^4}{p^3}$$

$$\Rightarrow \quad \rho' = \frac{r^4}{p^3}\times\frac{p^2r^4}{2r^6 - a^2b^2p^2} = \frac{r^8}{p(2r^6 - a^2b^2p^2)}.$$

In terms of original co-ordinates,

$$\rho' = \frac{p^8}{p'(2p^6 - a^2b^2p'^2)}.$$

But $p'r = p^2$ i.e. $p' = p^2/r$.

$$\therefore \quad \rho' = \frac{r.p^8}{p^2(2p^6 - a^2b^2p^4/r^2)} = \frac{rp^6\times r^2}{r^2.2p^6 - a^2b^2p^4}$$

$$= \frac{r^3p^6}{p^4(2r^2p^2 - a^2b^2)} = \frac{r^3p^2}{2r^2p^2 - a^2b^2}$$

$$= \frac{r^3}{2r^2 - \dfrac{a^2b^2}{p^2}} = \frac{r^3}{2r^2 - (a^2 + b^2 - r^2)}; \text{ from (1)}$$

$$= \frac{r^3}{3r^2 - (a^2 + b^2)}.$$

$\therefore$ The reqd. curvature $= \dfrac{1}{\rho'} = \dfrac{3r^2 - (a^2 + b^2)}{r^3} = \dfrac{3}{r} - \dfrac{a^2 + b^2}{r^3}$.

ANSWERS

6. $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

7. $(x + y)^{2/3} + (x - y)^{2/3} = (4a)^{2/3}$.

7

ASYMPTOTES

7.1. There are some well-known curves, e.g. circle or ellipse, which are closed curves and whose lengths are limited but there are also some curves whose graphs extend to infinity; for example parabola or hyperbola.

Let a curve be given and let a tangent be drawn at some point of the curve. If the point of contact of the tangent goes farther away from the origin, then the distance of the tangent from the origin will also go on changing; sometimes it will increase continuously and sometimes it will decrease continuously. But it may be possible that when the point of contact tends to infinity, then the tangent takes up a definite position of a st. line. This st. line is called the asymptote whose formal definition is given below:

Definition I—*A straight line at a finite distance from the origin to which a tangent to a curve tends as the point of contact tends to infinity, is called an asymptote of the curve.*

In simple language, an asymptote is a st. line which cuts a given curve in two points at infinity (i.e. touches at infinity) but is not itself at infinity. In other words, an asymptote is a tangent whose point of contact is $(x = \infty,\ y = \infty)$.

Under this definition, the equation of an asymptote of a given curve can be found out as follows:

Let the equation of the curve be $f(x, y) = 0$.

We suppose that the equation of the asymptote of this curve is $y = mx + c$.

We put $y = mx + c$ in the given curve so that we get a polynomial equation in x and thereby we find the abscissa of the points of intersection of the curve and the given st. line.

If this st. line $y = mx + c$ is the asymptote of the curve, then the two roots of this polynomial equation will be infinite.

From this, by equating the coeff. of the first two highest terms of x to zero, we find out the values of m and c and consequently by putting the values of m and c in $y = mx + c$ we get the equation of the asymptote.

We shall use the next article for finding out the condition for the two roots of the polynomial equation being infinite.

7.2. Infinite roots of an equation.

Let the equation of any polynomial be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0. \quad \ldots(1)$$

If a_n (const. term) = 0, $a_{n-1} = 0$, then

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 = 0$$

$$\Rightarrow \quad x^2\{a_0x^{n-2} + a_1x^{n-3} + \ldots + a_{n-2}\} = 0$$

so that $x^2 = 0$ i.e. two roots of x become zero. Thus we see that if we make the constant term and the coeff. of x in any polynomial equation each equal to zero, then the two roots of (1) become zero.

Now we put $x = \frac{1}{y}$ in (1), so that $y = \frac{1}{x}$.

It is evident that if $x = 0$, then $y = \infty$.

Putting $x = \frac{1}{y}$ in the equation (1), we get

$$a_0\frac{1}{y^n} + a_1\cdot\frac{1}{y^{n-1}} + a_2\cdot\frac{1}{y^{n-2}} + \ldots + a^{n-1}\cdot\frac{1}{y} + a_n = 0$$

$$\Rightarrow \quad a_ny^n + a_{n-1}y^{n-1} + \ldots + a_1y + a_0 = 0. \quad \ldots(2)$$

We have seen above that if $a_n = 0$, $a_{n-1} = 0$, then the two values of x will be zero i.e. the two values of y will be ∞; a_n is the coeff. of the highest term in the equation (2) and a_{n-1} is the coeff. of the next highest term.

Thus we conclude that if the coeffs. of first two highest terms of the equation (2) be zero, then the two roots of the equation (2) will be infinite.

Similarly it can be shown that if the coeffs. of first three highest terms of the equation (2) {or any polynomial equation} be zero, then the three roots of the equation (2) will be infinite and so on.

Thus if the two roots of any polynomial equation be infinite, then we shall have to equate the coeffs. of the first two highest terms of the given polynomial equation to zero.

We whall use the above consideration in finding out the asymptotes of any curve.

EXAMPLE : *Find the asymptotes of* $x^3 + y^3 = 3axy$.

SOLUTION : Let the equation of the asymptote be $y = mx + c$.

We put $y = mx + c$ in the equation of the curve $x^3 + y^3 = 3axy$.

Then $x^3 + (mx + c)^3 - 3ax(mx + c) = 0$

$\Rightarrow$ $x^3 + (m^3x^3 + 3m^2x^2.c + 3mx.c^2 + c^3) - 3amx^2 - 3acx = 0$

$\Rightarrow$ $x^3(1 + m^3) + x^2(3m^2c - 3am) + x(3mc^2 - 3ac) + c^3 = 0.$ ...(1)

If the two roots of the equation (1) be infinite, then from the last article

$$m^3 + 1 = 0 \quad \text{...(2)}$$

and $3m^2c - 3am = 0 \Rightarrow 3m(cm - a) = 0.$ ...(3)

From (2), $(m + 1)(m^2 - m + 1) = 0.$

$\therefore$ $m = -1$; the other two roots are imaginary.

From (3), $cm - a = 0 \quad \therefore \quad c = \dfrac{a}{m} = \dfrac{a}{-1} = -a.$

Now, putting the values of m and c, the equation of the real asymptote is

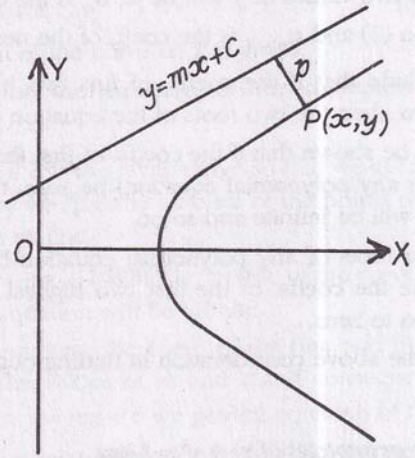
$$y = -1.x - a \Rightarrow x + y + a = 0.$$

Definition II—The other definition of asymptote can be given as follows:

If $y = mx + c$ is an asymptote to a curve, then

$$m = \underset{x \to \infty}{Lt} \frac{y}{x} \text{ and } c = \underset{x \to \infty}{Lt} (y - mx).$$

Let $y = mx + c$ be an asymptote of the curve where m and c are to be determined.



Let $P(x, y)$ be any point on the curve. From P, draw a perpendicular on $y = mx + c$ whose length is $= p$.

Then $$p = \frac{y - mx - c}{\sqrt{1 + m^2}}. \qquad ...(1)$$

From the figure, it is obvious that as the point tends to infinity along the curve, the distance between the curve and the line becomes lesser and lesser i.e. $p \to 0$; i.e. when $y = mx + c$ touches the curve at infinity, then $p \to 0$.

Thus when $x \to \infty$, then $p \to 0$. From this, we get the values of m and c.

Now taking the limit of (1) when $x \to \infty$,

$$Lt\, p = Lt \frac{y - mx - c}{\sqrt{1 + m^2}} \to 0 = Lt\,(y - mx - c).$$

$$\therefore \quad c = \underset{x \to \infty}{Lt}\,(y - mx). \qquad ...(2)$$

Again, from $y = mx + c + p\sqrt{1 + m^2}$,

$$\frac{y}{x} = m + \frac{c}{x} + \frac{p}{x}\sqrt{1 + m^2}$$

$$\therefore \quad \underset{x \to \infty}{Lt} \frac{y}{x} = m + \underset{x \to \infty}{Lt} \frac{c}{x} + \underset{x \to \infty}{Lt} \frac{p}{x} \sqrt{1 + m^2}.$$

$$\therefore \quad m = \underset{x \to \infty}{Lt} \frac{y}{x}. \qquad ...(3)$$

Thus we find out the values of c and m from (2) and (3) and thereby the equation of the asymptote $y = mx + c$ is found out.

We again find out the equation of the asymptote of the last example by this method.

EXAMPLE : *Find the asymptotes of the curve* $x^3 + y^3 = 3axy$.

SOLUTION : Let $y = mx + c$ be the equation of the asymptote.

Dividing the equation of the given curve by x^3, we get

$$1 + \left(\frac{y}{x}\right)^3 = 3a \cdot \frac{y}{x^2} = 3a \cdot \frac{y}{x} \cdot \frac{1}{x}.$$

$\therefore$ Taking the limit when $x \to \infty$, we get

$$1 + m^3 = 3am.0 = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - m + 1) = 0.$$

$\therefore$ the real value of $m = -1$.

Now $$c = \underset{x \to \infty}{Lt}\,(y - mx) = \underset{x \to \infty}{Lt}\,(y + x).$$

Therefore we shall put $c-x$ for y in the equation of the given curve and then take the limit when $x \to \infty$.

Thus we find that

$$x^3 + (c - x)^3 = 3ax(c - x)$$

$$\Rightarrow \quad x^3 + c^3 - 3c^2x + 3cx^2 - x^3 = 3acx - 3ax^2$$

$$\Rightarrow \quad 3cx^2 - 3c^2x + c^3 - 3acx + 3ax^2 = 0$$

$$\Rightarrow \quad (3c + 3a)x^2 - 3cx(c + a) + c^3 = 0$$

$$\Rightarrow \quad (3c + 3a) - 3c(c + a)\frac{1}{x} + \frac{c^3}{x^2} = 0.$$

$\therefore$ Taking the limit when $x \to \infty$, we have

$$3c + 3a = 0 \quad \therefore \quad c = -a.$$

$\therefore$ The reqd. equation of the asymptote, as before, is

$$y = -x - a \Rightarrow x + y + a = 0.$$

Note : We shall establish the correspondence between the two definitions later on. But it is easy to see that the second method is not helpful in all cases. Therefore we shall always use the first method. But the first method can be further simplified so that it can be used as a formula.

We proceed like this :

If the equation of the curve be $f(x, y) = 0$, then we collect the terms of similar powers in x and y together so that the given equation is expressed as the sum of the terms of similar powers in x and y.

Let the equation of the curve be

$$y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0.$$

Then

terms of the 3rd degree will be $y^3 - 3x^2y + xy^2 - 3x^3$

" 2nd " $2y^2 + 2xy$

" 1st " $4x + 5y$.

In this formation, the first collection is a homogeneous function of 3rd degree in x and y; the second collection is a homogeneous function of 2nd degree in x and y; the third collection is a homogeneous function of 1st degree in x and y.

Consequently we can express the first collection as $x^3\phi\left(\frac{y}{x}\right)$, the second collection as $x^2\phi\left(\frac{y}{x}\right)$ and the third collection by $x\phi\left(\frac{y}{x}\right)$.

By writing the equation of the curve in this fashion, we easily get a formula for finding out its asymptotes. The method is described in the following article.

7.3. Asymptotes of an algebraic curve

Let the equation of the curve be

$$(a_0x^n + a_1x^{n-1}y + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \ldots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-3}y + \ldots + c_ny^{n-2}) + \ldots = 0.$$

In the first bracket we put all those terms in which the sum of the indices of x and y is n; i.e. the first bracket is a homogeneous function in x and y of degree n.

Similarly the second bracket is a homogeneous function of degree $(n-1)$ and so on.

Hence we can write the equation of the curve as follows:

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots = 0. \qquad \ldots(1)$$

where $x^r\phi_r\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree r.

Let the equation of the asymptote be $y = mx + c$. $\qquad \ldots(2)$

Now we want to find out the point of intersection of the line $y = mx + c$, with the given curve.

From the given st. line $y = mx + c$, we have

$$\frac{y}{x} = m + \frac{c}{x}. \qquad \ldots(3)$$

We substitute this value of (3) in (1). Therefore

$$x^n\phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + x^{n-2}\phi_{n-2}\left(m + \frac{c}{x}\right) + \ldots = 0 \qquad \ldots(4)$$

from which the abscissae of the points of intersection of the st. line and the curve are found out.

Expanding (4) by Taylor's Theorem, we get

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{x^2}\cdot\frac{1}{2!}\phi_n''(m) + \ldots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi_{n-1}'(m) + \ldots\right\} + x^{n-2}\left\{\phi_{n-2}(m) + \frac{c}{x}\phi_{n-2}'(m) + \ldots\right\} + \ldots = 0.$$

$$\Rightarrow \quad x^n\phi_n(m) + x^{n-1}\{c\,\phi_n'(m) + \phi_{n-1}(m)\}$$

$$+ x^{n-2}\left\{\frac{c^2}{2!}\,\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m)\right\} + \ldots = 0. \quad \ldots(5)$$

The equation (5) is an equation of nth degree which means that a st. line cuts a curve of nth degree in n points, real or imaginary.

If the st. line (2), is an asymptote of the curve, then its two points of intersection will be at infinity.

Therefore the two roots of the equation (5) will be infinite.

Hence the coeffs. of two highest terms of the equation (5) i.e. the coeffs of x^n and x^{n-1} are each = 0.

$$\therefore \qquad \phi_n(m) = 0 \qquad \ldots(6)$$

and $c\phi_n'(m) + \phi_{n-1}(m) = 0$

$$\therefore \qquad c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} \qquad \ldots(7)$$

Let the n roots of the equation (6) be $m_1, m_2, m_3, \ldots m_n$. (here it is assumed that an equation of nth degree has n roots). Then putting successively $m = m_1, m_2, m_3 \ldots m_n$ in (7), we get the values of c, say, $c = c_1, c_2, c_3, \ldots c_n$.

Thus by putting successivley the values of m and the corresponding values of c in (2), we shall get the following equations of the asymptotes:

$$y = m_1x + c_1$$

$$y = m_2x + c_2$$

$$y = m_3x + c_3$$

.....

$$y = m_nx + c_n.$$

Note : Sometimes the value of c cannot be found out by (7). This is the case when c is of the form $\frac{0}{0}$. In this case we find out the value of c by equating to zero the coeff. of x^{n-2} i.e. the coeff. of the term of the third highest degree i.e. we shall get the value of c from the equation

$$\frac{c^2}{2!}\,\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0.$$

In particular, if $n = 3$ i.e. if there be a curve of third degree, then putting

$n = 3$ in (6), (7) we get the values of m and c from the equations

$$\phi_3(m) = 0$$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)}.$$

If we do not get the value of c from $-\frac{\phi_2(m)}{\phi_3'(m)}$ e.g.; when $c = \frac{0}{0}$, then in that case the value of c is found out from the equation

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0.$$

7.4. Working rule for finding the asymptotes

(i) We shall get $\phi_n(m)$ by putting $x = 1$ and $y = m$ in the terms of the highest degree i.e. in $x^n\phi_n\left(\frac{y}{x}\right)$.

Now we shall put $\phi_n(m) = 0$ and solve it to get say, $m = m_1, m_2, m_3, \ldots m_n$.

(ii) We shall get $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the terms of the second highest degree.

Then we shall respectively get the values $c_1, c_2, c_3, \ldots c_n$ of c by putting successively $m = m_1, m_2, m_3, \ldots m_n$ in the formula $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$.

(iii) Sometimes, for a certain value of m, c assumes the form $\frac{0}{0}$. In that case, the value of c is found out by using the formula

$$\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-1}(m) = 0.$$

(iv) Now for $m = m_1, m_2, \ldots m_n$ and the corresponding values $c = c_1, c_2, c_3, \ldots c_n$ of c, the equations of the asymptotes will be

$$y = m_1x + c_1$$

$$y = m_2x + c_2$$

..... etc.

7.5. The case of parallel asymptotes

We have found out the following two equations in the last article for finding out the values of m and c.

$$\phi_n(m) = 0 \qquad \ldots(1)$$

$$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \qquad \ldots(2)$$

Let the roots of the equation $\phi_n(m) = 0$ be $m_1, m_2, m_3, \ldots m_n$.

We know from Theory of equations that if the two roots of $\phi_n(m) = 0$ be equal, say $m_1 = m_2$, then $\phi_n'(m_1) = 0$.

Case I. If $m_1, m_2, m_3, \ldots m_n$ are different from each other, then $\phi_n'(m) \neq 0$ for any root m.

In this case c will be determined from (2).

Case II. Let the two roots of $\phi_n(m) = 0$ be equal say $m_1 = m_2$, then $\phi_n'(m_1) = 0$. In this case if $\phi_{n-1}(m_1) \neq 0$, then from (2), c becomes infinite. Hence the line $y = m_1x + c_1$, in this case, cuts an infinite intercept from the y-axis and thus it wholly lies at infinity. Hence $y = m_1x + c_1$ is not an asymptote in this case. If the two roots of $\phi_n(m) = 0$ be equal, say $m_1 = m_2$, then $\phi_n'(m_1) = 0$. Also, if $\phi_{n-1}(m_1) = 0$, then whatever be the value of c,

$$c\phi_n'(m) + \phi_{n-1}(m) = 0 \text{ identically.}$$

Thus $y = mx + c$ cuts the curve at two points at infinity, whatever be the value of c. The value of c cannot be found out from this. Here, the value of c, so far is unknown, and its value is at our disposal, Therefore we choose the value of c in such a manner that the coeff. of the succeeding term i.e. the coeff. of the term of the third highest degree viz.,

$$\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0. \qquad \ldots(3)$$

This is a quadratic equation in c from which we get two values of c.

Thus corresponding to a particular value of m, say $m = m_k$, we get two values of c i.e. c_{k_1}, c_{k_2}. Therefore there would be two asymptotes corresponding to $m = m_k$:

$$y = m_kx + c_{k_1}$$

and $$y = m_kx + c_{k_2}.$$

This is obvious that these two asymptotes are parallel and each one of these cuts the curve at three points at infinity (because the coeffs. of the first three highest degree terms have been made zero).

7.6. Total number of asymptotes of a curve

Let $y = mx + c$ be the equation of an asymptote. We know that the value of m is found out by solving the equation $\phi_n(m) = 0$. Since the equation of nth degree has n roots, therefore we shall get n values of m by solving $\phi_n(m) = 0$. We shall get an asymptote corresponding to each value of m. Hence the curve of nth degree has generally n asymptotes, real or imaginary.

Worked out Examples

EXAMPLE 1. *Find the real asymptotes of the curve* $x^3 + y^3 = 3axy$.

(P.U. 46S, 50S, 69S, 87; Bhag. 62S, 72A, 85; R.U. 62A; Mithila 77; M.U. 72S, 74A, 76, 80, 82, 83H, 84; B.U. 86)

SOLUTION : Given the equation of the curve $x^3 + y^3 - 3axy = 0$ in which the highest degree $= 3$.

Therefore putting $x = 1$ and $y = m$ in the third degree terms and equating them to zero, we get

$$\phi_3(m) = 1 + m^3 = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = -1 \text{ or } m = \frac{1 \pm i\sqrt{3}}{2}.$$

Again $\phi_2(m) = -3am$.

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3am}{3m^2} = \frac{a}{m}.$$

If $m = -1$, then $c = -a$.

The other two values of m are imaginary, therefore the two asymptotes are imaginary. Thus only one asymptote is real and its equation will be

$$y = -1.x + (-a)$$

$$\Rightarrow \quad y = -x - a \Rightarrow x + y + a = 0.$$

EXAMPLE 2. *Find the asymptotes of the curve*

$$x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0.$$

(P.U. 66A; B.U. 57A; Bhag. 65A, 69A; M.U. 86)

SOLUTION : Putting $x = 1$, $y = m$ in the third degree terms of the given curve and equating them equal to zero, we get

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2 = 0$$

$$\Rightarrow \quad (1 + 2m) - m^2(1 + 2m) = 0 \quad \Rightarrow \quad (1 + 2m)(1 - m^2) = 0$$

$$\Rightarrow \quad (1 + 2m)(1 - m)(1 + m) = 0. \quad \therefore \quad m = 1, -1, -\frac{1}{2}.$$

Again, $\phi_2(m) = m - m^2 = m(1 - m)$.

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{m(1-m)}{-6m^2 + 2 - 2m}.$$

When $m = 1$, then $c = 0$.

When $m = -1$, then $c = -\dfrac{(-1)(1+1)}{-6.1 + 2 + 2} = -\dfrac{-2}{-2} = -1$.

When $m = -\dfrac{1}{2}$, then $c = -\dfrac{-\frac{1}{2}\left(1 + \frac{1}{2}\right)}{-6 \cdot \frac{1}{4} + 2 + 2 \cdot \frac{1}{2}} = \dfrac{\frac{1}{2}\left(\frac{3}{2}\right)}{-\frac{3}{2} + 3}$

$$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}.$$

$\therefore$ Putting the corresponding values of m and c, the equations of the asymptotes are

$$y = 1.x + 0 \quad \Rightarrow \quad y = x$$

$$y = -1.x - 1 \quad \Rightarrow \quad x + y + 1 = 0$$

$$y = -\frac{1}{2}.x + \frac{1}{2} \quad \Rightarrow \quad x + 2y - 1 = 0.$$

EXAMPLE 3. *Find the asymptotes of*

$$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0.$$

(M.U. 63A, 65A; P.U. 94H)

SOLUTION : Putting $x = 1$, $y = m$ in the third degree terms of the given curve and equating them equal to zero, we get,

$$\phi_3(m) = 1 + m - m^2 - m^3 = 0$$

$$\Rightarrow \quad (1 + m) - m^2(1 + m) = 0 \quad \Rightarrow \quad (1 + m)(1 - m^2) = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - 1) = 0 \quad \Rightarrow \quad (m + 1)(m + 1)(m - 1) = 0.$$

$$\therefore \quad m = 1, -1, -1.$$

Again $\phi_2(m) = 2m + 2m^2 = 2m(m + 1)$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m(m+1)}{1 - 2m - 3m^2}.$$

When $m = 1$, then $c = -\dfrac{2.2}{1 - 2 - 3} = -\dfrac{4}{-4} = 1$.

When $m = -1$, then $c = \dfrac{0}{0}$.

We, therefore, find the value of c by using

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}(-2 - 6m) + c(2 + 4m) + (-3 + m) = 0$$

$$\Rightarrow \quad -c^2(1 + 3m) + c(2 + 4m) + (-3 + m) = 0$$

$$\Rightarrow \quad -c^2(1 - 3) + c(2 - 4) + (-3 - 1) = 0$$

$$\Rightarrow \quad -c^2(-2) + c(-2) + (-4) = 0$$

$$\Rightarrow \quad c^2 - c - 2 = 0 \Rightarrow (c + 1)(c - 2) = 0.$$

$$\therefore \quad c = -1, 2.$$

Hence when $m = 1$, then $c = 1$.

When $m = -1$, then $c = -1$ or 2.

$\therefore$ The equations of asymptotes are

$y = 1.x + 1$ i.e. $y = x + 1$

$y = -1.x + (-1)$ " $y + x + 1 = 0$

$y = -1.x + 2$ " $y + x - 2 = 0$.

EXAMPLE 4. *Find the asymptotes of* $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$. **(P.U. 65A)**

SOLUTION : Here the highest degree of the given curve is 3.

Therefore putting $x = 1$ and $y = m$ in the 3rd degree terms and equating them equal to zero, we get

$$\phi_3(m) = 4 - 3m^2 - m^3 = 0$$

$$\Rightarrow \quad m^3 + 3m^2 - 4 = 0 \quad \Rightarrow \quad (m - 1)(m^2 + 4m + 4) = 0$$

$$\Rightarrow \quad (m - 1)(m + 2)^2 = 0$$

$$\therefore \quad m = 1 \text{ or } -2 \text{ or } -2.$$

Again, $\phi_2(m) = 2 - m - m^2$

$$= -(m^2 + m - 2) = -(m - 1)(m + 2).$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-(m - 1)(m + 2)}{-6m - 3m^2}$$

$$= -\frac{(m - 1)(m + 2)}{3m(m + 2)} = -\frac{m - 1}{3m}, \text{ provided } m + 2 \neq 0.$$

When $m = 1$, then $c = -\dfrac{1-1}{3} = 0$.

When $m = -2$, then $c = \dfrac{0}{0}$.

$\therefore$ We use $\dfrac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$.

Hence $\dfrac{c^2}{2}(-6 - 6m) + c(-1 - 2m) + (0) = 0$

$\Rightarrow$ $-3c^2(1 + m) - c(1 + 2m) = 0$

$\Rightarrow$ $3c^2(1 + m) + c(1 + 2m) = 0$

$\Rightarrow$ $c\{3c(1 + m) + (1 + 2m)\} = 0$

$\therefore$ $c = 0$ or $3c(1 + m) + (1 + 2m) = 0$

$\Rightarrow$ $3c(1 - 2) + (1 - 4) = 0$; putting $m = -2$

$\Rightarrow$ $-3c - 3 = 0$. $\therefore$ $c = -1$.

Thus when $m = 1$, $c = 0$

$m = -2$, then $c = 0$ or $c = -1$.

$\therefore$ The equations of asymptotes are

$y = x \quad \Rightarrow \quad y - x = 0$

$y = -2x + 0 \quad \Rightarrow \quad y + 2x = 0$

$y = -2x - 1 \quad \Rightarrow \quad y + 2x + 1 = 0.$

7.7. Asymptotes parallel to the axes for an algebraic curve.

Let the equation of the curve be

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \ldots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-3}y + \ldots + c_ny^{n-2}) + \ldots = 0$$

Arranging the above equation in descending powers of x, we get

$$a_0x^n + x^{n-1}(a_1y + b_1) + x^{n-2}(a_2y^2 + b_2y + c_2) + \ldots + a_ny^n = 0. \quad \ldots(1)$$

Hence if $a_0 = 0$ and the value of y be so chosen that $a_1y + b_1 = 0$, we find that the coeffs. of terms of the first two highest indices in x in the equation (1) i.e., the coeffs. of x^n and x^{n-1} become zero. That is, if the st. line $a_1y + b_1 = 0$ be made to cut the curve, then the two roots of (1) become infinite, i.e. the st. line $a_1y + b_1 = 0$ i.e. $y = -\dfrac{b_1}{a_1}$ becomes the asymptote. But

$y = -\frac{b_1}{a_1}$ is the equation of the st. line parallel to x-axis. Thus if any curve of nth degree be given and we are required to find the equation of the asymptote parallel to the x-axis, then we should equate to zero, the coeffs. of x^n and x^{n-1} Similarly by arranging the equation of the curve in descending powers of y, we find that

$$a_n y^n + (a_{n-1}x + b_n)y^{n-1} + (a_{n-1}x^2 + b_{n-1}x + c_n)y^{n-2} + \ldots + a_0 x^n = 0. \quad \ldots(2)$$

Hence if $a_n = 0$ and x be so chosen that $a_{n-1}x + b_n = 0$, then the coeffs. of the first two highest powers of y in the equation (2) become zero. Therefore $a_{n-1}x + b_n = 0$ i.e. $x = -\frac{b_n}{a_{n-1}}$ becomes the asymptote which is parallel to y-axis. Therefore if any curve of nth degree be given and we are required to find the equation of the asymptote parallel to y-axis, then we should put to zero the coeffs. of y^n and y^{n-1}. Again., if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$ in the equation (1) and y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, then the three roots of (1) will be infinite, i.e. these pair of lines will cut the curve (1) at three points at infinity. Therefore the lines $a_2y^2 + b_2y + c_2 = 0$ represent those pair of asymptotes, real or imaginary, which are parallel to x-axis.

Similarly if we put $a_n = 0$, $a_{n-1} = 0$, $b_n = 0$ in the equation (2) and if x be so chosen that $a_{n-2}x^2 + b_{n-1}x + c_n = 0$, then the three roots of the equation (2) will be infinite. Hence the lines $a_{n-2}x^2 + b_{n-1}x + c_n = 0$ represent the equation of two asymptotes, real or imaginary which are parallel to y-axis.

7.8. Working rule

(i) Let the equation of the curve in x and y be of the nth degree. If we are required to find the equation of the asymptote parallel to x-axis, then we should put to zero, the coeffs. of x^n and x^{n-1}. If the term involving x^n is absent, then the equation of the asymptote parallel to x-axis is found out simply by equating to zero, the coeff. of x^{n-1}.

(ii) If the terms containing x^n and x^{n-1} are absent, then the equation of the two asymptotes parallel to x-axis is found out by equating to zero the coeff. of x^{n-2}.

(iii) If we are required to find out the equation of asymptotes parallel to y-axis, then we shall put to zero the coeff. of y^n and y^{n-1}. If the term containing y^n is absent, then the equation of the asymptote is found out simply by equating to zero the coeff. of y^{n-1}.

(iv) If the terms containing y^n and y^{n-1} are both absent, then the equation of the two asymptotes parallel to y-axis is found out by equating to zero the coeff. of y^{n-2}.

Note : If the equation of the curve be of nth degree and if the coeff. of x^n i.e. $a_0 \neq 0$, then we shall not get any asymptote parallel to x-axis or if the coeff. of y^n i.e. $a_n \neq 0$, then we shall not get any asymptote parallel to y-axis.

For example, we consider the curve $x^3 + y^3 = 3axy$. This is the equation of 3rd degree. In this, the coeff. of x^3 is = 1 ($\neq 0$). Therefore here we shall not get any asymptote parallel to x-axis. Again, the coeff. of y^3 is =1 ($\neq 0$). Therefore we shall not get any asymptote parallel to y-axis.

Hence the given curve does not possess any asymptote parallel to x-axis or y-axis.

Worked out Examples

EXAMPLE 1. *Find all the asymptotes of the curve* $y^2(x^2 - a^2) = x$.

SOLUTION : This is the curve of 4th degree.

Here the coeff. of the highest power in y i.e. the coeff. of y^2 is $x^2 - a^2$.

$\therefore$ The equation of asymptotes parallel to y-axis will be $x^2 - a^2 = 0$ i.e. $x = \pm a$.

Similarly the coeff. of the highest power in x i.e. the coeff. of x^2 is y^2.

$\therefore$ The equation of asymptote parallel to x-axis will be $y^2 = 0$ i.e. $y = 0$.

Thus the asymptotes of the curve will be

$$x - a = 0,\ x + a = 0 \text{ and } y = 0.$$

EXAMPLE 2. *Find all the asymptotes of the curve* $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$.

(P.U. 45S; B.U. 85)

SOLUTION : The given equation can be written as follows :

$$a^2y^2 + b^2x^2 = x^2y^2.$$

This is the curve of 4th degree in which the terms involving x^4, x^3 and y^4, y^3 are absent. Now, here the coeff. of the highest power in x i.e. the coeff. of x^2 is $b^2 - y^2$.

$\therefore$ The equation of asymptotes parallel to x-axis will be $b^2 - y^2 = 0$ i.e. $y = \pm b$.

Similarly the coeff. of highest power in y i.e. the coeff. of y^2 is $a^2 - x^2$.

$\therefore$ The equation of asymptotes parallel to y-axis will be $a^2 - x^2 = 0$ i.e. $x = \pm a$.

Thus the reqd. equation of the asymptotes will be $x = \pm a$, $y = \pm b$.

EXAMPLE 3. *Find the asymptotes of the curve*

$$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0.$$

(P.U. 32S, 49A, 51S, 55A; B.U. 68A, 74A; R.U. 64A, 65A, 76A, 81)

SOLUTION : Here the highest power term in x is x^3 and the coeff. of x^3 is $1 (\neq 0)$.

$\therefore$ There is no asymptote parallel to x-axis.

Again, the highest power term in y is $xy^2 = 0$.

We shall get one asymptote $x = 0$ from the coeff. x of y^2.

For finding out the equation of the other two remaining asymptotes, we follow the previous method.

Here the highest degree of the given equation is 3. Therefore putting $x = 1$ and $y = m$ in the 3rd degree terms, we get

$$\phi_3(m) = 1 - 2m + m^2 = 0$$

$$\Rightarrow \quad (m-1)^2 = 0 \quad \therefore \quad m = 1.$$

Again $\phi_2(m) = 1 - m$.

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{1-m}{2m-2} = \frac{1-m}{2(m-1)}.$$

When $m = 1$, then $c = \frac{0}{0}$.

$\therefore$ We use $\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$.

Hence $\frac{c^2}{2}(2) + c(-1) + 0 = 0$

$$\Rightarrow \quad c^2 - c = 0 \quad \Rightarrow \quad c(c-1) = 0$$

$$\therefore \quad c = 0,\ c = 1.$$

Thus when $m = 1$, then $c = 0$ or $c = 1$.

$\therefore$ The equations of asymptotes are

$$y = 1.x + 0 \quad \Rightarrow \quad y = x$$

$$y = 1.x + 1 \quad \Rightarrow \quad y = x + 1$$

and $x = 0$ (found out before).

EXAMPLE 4. *Find the asymptotes of the curve* $y^2(x - 2a) = x^3 - a^3$.

(P.U. 46A; B.U. 58S, 65A, 78)

SOLUTION : The equation of the given curve is of 3rd degree. Here the coeff. of x^3 is $1 \neq 0$, therefore there is no asymptote parallel to x-axis. Again, the coeff. of $y^3 = 0$, therefore equating to zero the coeff. of y^2, we get $x - 2a = 0$. Therefore the equation of the asymptote parallel to y—axis is $x - 2a = 0$.

Now, putting $x = 1$, $y = m$ in the 3rd degree term of the given equation and equating that to zero, we get

$$\phi_3(m) = m^2 - 1 = 0$$

$$\Rightarrow \quad (m - 1)(m + 1) = 0, \quad \therefore \quad m = 1, -1.$$

Again, $\phi_2(m) = -2am^2$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-2am^2}{2m} = am.$$

$\therefore$ When $m = 1$, then $c = a$,

when $m = -1$, then $c = -a$.

Thus putting the values of m and c, the required asymptotes are

$$y = 1.x + a \quad \text{i.e.} \quad y = x + a$$

$$y = -1.x - a \quad \text{"} \quad y + x + a = 0$$

and $x - 2a = 0$.

EXAMPLE 30

Find the asymptotes of the following :

1. $x^3 + y^3 - 6xy = 0$ **(P.U. 31A; M.U. 78)**
2. $x^2y + xy^2 = a^3$ **(B.U. 57S; R.U. 71S)**
3. $x^2y = x^3 + x + y$ **(P.U. 44S)**
4. $x^2y^2 = a^2(x^2 + y^2)$ **(B.U. 69A)**
5. $x^2y^2 = a^2y^2 + b^2x^2$ **(P.U. 45S)**
6. $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

7. $x^4 - y^4 + 2ax^2y = b^2x^2$ **(P.U. 39A)**

8. $9x^4 - 4x^2y^2 + x^2 + y^2 = 1$ **(P.U. 65S)**

9. $y^2(x - 2a) = x^2(x - a)$ **(R.U. 80)**

10. $xy(x - y) + bx^2 - ay^2 = 0$ **(P.U. 41S; B.U. 67S)**

11. $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$

12. $\frac{y^2}{x^2} = \frac{(a - x)^2}{a^2 + x^2}$

13. $(y - a)^2(x^2 - a^2) = x^4 + a^4$

14. $(x^2 - 2ax)(x^2 + y^2) = b^2x^2$ **(P.U. 55S)**

15. $(a + x)^2(b^2 + x^2) = x^2y^2$ **(P.U. 56A, 58A)**

16. $y^3 + x^2y + 2xy^2 - y + 1 = 0$
(P.U. 61A; M.U. 68A; R.U. 63A, 70A, 82; B.U. 77)

17. $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$
(P.U. 51, 55, 62S; B.U. 58A, 68A, 74; R.U. 65, 76)

18. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ **(P.U. 60A; R.U. 67A; B.U. 72A)**

19. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$ **(R.U. 72A)**

20. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$ **(P.U. 47A, 50A, 61A; B.U. 56S, 80; Bhag. 63S, 69S; R.U. 68S, 69S, 70S, 75A)**

21. $2y^3 + 3y^2x + 3yx^2 + x^3 - y^2 - x^2 - x = 0$ **(P.U. 58S, 92H)**

22. $x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 1 = 0$
(P.U. 57S, 70S, 79; B.U. 61A; R.U. 69A, 73A)

23. $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$
(B.U. 57A; Bhag. 65A, 69A; P.U. 66A, 76A)

24. $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0$ **(P.U. 37S, 52A, 63A)**

25. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$ **(P.U. 53S, 54S; Bhag. 88)**

26. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0$
(B.U. 55A; R.U. 66S, 72S)

27. $x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0$
(P.U. 63A, 65A, 90H; B.U. 67A; R.U. 72H)

28. $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$

29. $y^3 - 3x^2y - 2x^3 + 7y^2 + 5xy - 2x^2 + 3y - 3x - 1 = 0$
(B.U. 60S, 64S; M.U. 66S)

30. $y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0$
(P.U. 58A; M.U. 70A)

31. $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$ **(P.U. 63S)**

32. $2x^3 - x^2y - 2xy^2 + y^3 - 4y^2 + 12xy - 8x^2 + 3y - 3x + 4 = 0$ **(B.U. 53A; Bhag. 62A)**

33. $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$ **(P.U. 53A)**

34. $x(y - 3)^3 = 4y(x - 1)^3$ **(P.U. 56S, 71A)**

35. $2x(y - 5)^2 = 3(y - 2)(x - 1)^2$ **(Bhag. 66; M.U. 81)**

36. $(x^2 - y^2)^2 - 4y^2 + y = 0$

37. $(x^2 - y^2)y - 2y^2 + 5x - 7 = 0$ **(Bhag. 65; M.U. 66A, 77; R.U. 84)**

38. $(y^2 - x^2)^2 - 2(x^2 + y^2) - 1 = 0$ **(P.U. 47S; B.U. 54S)**

39. $(x^2 - y^2)^2 - 2a^2(x^2 + y^2) + a^3x = a^4$ **(P.U. 54A; B.U. 59A)**

40. $(x + y)^2(x^2 + y^2 + xy) = a^2y^2 + a^3(x - y)$

ANSWERS

1. $x + y + 2 = 0$ is the only real asymptote.

2. $x = 0,\ y = 0,\ x + y = 0$ **3.** $y = x,\ x = \pm 1$

4. $x = \pm a,\ y = \pm a$ **5.** $x = \pm a,\ y = \pm b$

6. $x = \pm a$, the other asymptotes are imaginary.

7. $2x - 2y + a = 0,\ 2x + 2y - a = 0$ are the only real asymptotes

8. $2x = \pm 1,\ 2y - 3x = 0,\ 2y + 3x = 0.$

9. $x - 2a = 0,\ 2y - 2x - a = 0;\ 2x + 2y + a = 0;$

10. $x - y + b - a = 0,\ x + a = 0,\ y + b = 0.$

11. $x = \pm a,\ y = x,\ y = -x.$

12. $y - x + a = 0,\ y + x - a = 0$ are the only real asymptotes.

13. $x = \pm a,\ y \pm x = a.$

14. $x = 0,\ x - 2a = 0$, the other asymptotes are imaginary.

15. $x = 0,\ x - y + a = 0,\ x + y + a = 0.$

16. $y = 0,\ x + y = 1,\ x + y + 1 = 0.$ **17.** $x = 0,\ y + x = 0,\ y + x = 1.$

18. $y = x,\ 2y + x \pm 1 = 0.$ **19.** $x = 0,\ y = 0,\ x = 1,\ y = 1.$

20. $x + y = 0,\ y - x = 0,\ y - x - 1 = 0.$ **21.** $x + 2y = \frac{5}{3}$

22. $x - 2y = 0,\ x + 2y + 2 = 0,\ x + 2y - 2 = 0$

23. $x - y = 3,\ x + y + 1 = 0,\ x + 2y - 1 = 0.$

24. $x - y = 0,\ x + y = 0,\ x + 2y + 1 = 0.$

25. $x - y + 1 = 0$, $x + y = 1$, $x + 2y = 0$.

26. $x + 3 = 0$, $x = 2y$, $x + 2y = 6$.

27. $2x + 5 = 0$, $2x + 4y + 1 = 0$, $x + 2y + 2 = 0$.

28. $y + x = 2$, $y = x + 2$, $y = 2x - 4$.

29. $y + x + 1 = 0$, $y + x + 2 = 0$, $y - 2x + 4 = 0$.

30. $x + y = 0$, $y + \sqrt{3}x + 1 = 0$, $y - \sqrt{3}x + 1 = 0$.

31. $x + y = 1$, $y = x \pm (3 + \sqrt{5})/2$.

32. $x - y = 0$, $2x - y = 0$, $x + y - 4 = 0$.

33. $x = \pm a$, $y = \pm a$.

34. $x = 0$, $y = 0$, $2y = 4x + 3$, $2y + 4x = 15$.

35. $x = 0$, $y = 2$, $2y - 3x - 10 = 0$.

36. $x - y = \pm 1$, $x + y = \pm 1$.

37. $y = 0$, $x - y - 1 = 0$, $x + y + 1 = 0$.

38. $x - y + 1 = 0$, $x - y - 1 = 0$, $x + y = 1$, $x + y + 1 = 0$.

39. $x + y - a = 0$, $x + y + a = 0$, $x - y - a = 0$, $x - y + a = 0$.

40. $x + y - a = 0$, $x + y + a = 0$.

7.9. Miscellaneous Methods

We have given above, the general method for finding out the asymptote of any algebraic curve. But there are certain curves whose equations are of special forms and hence their asymptotes can be found out easilv. They are the following:

Method I. *By inspection*

Form; $P_n + Q_{n-2} = 0$.

When the equation of the curve can be put in the form $P_n + Q_{n-2} = 0$, where P_n is a function of nth degree in x and y, and P_n can be expressed in the form of a product of n linear factors in x and y (and none of linear factors is repeated) and Q_{n-2} is a function of $(n-2)$th degree in x and y, then all the asymptotes of the curve can be obtained by $P_n = 0$.

Let $P_n \equiv (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) \ldots (a_nx + b_ny + c_n)$.

If we put $a_1x + b_1y + c_1 = 0$ in the equation $P_n + Q_{n-2} = 0$ of the curve, the equation of the curve becomes $Q_{n-2} = 0$ i.e. the coeffs. of x^n (or y^n) and x^{n-1} (or y^{n-1}) become zero.

$\therefore$ $a_1x + b_1y + c_1 = 0$ is the equation of one asymptote.

Similarly $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$, ... $a_nx + b_ny + c_n = 0$ are the other asymptotes of the curve.

Thus the n asymptotes of the curve are given by $P_n = 0$.

EXAMPLE : *Find the asymptotes of the curve*

$$(x - y)(x + y + 1)(2x + y + 1) + x + y - 1 = 0.$$

SOLUTION : The equation to the curve is of 3rd degree in x and y.

Now if in the equation of the curve $x - y = 0$

or $x + y + 1 = 0$ or $2x + y + 1 = 0$,

then the equation of the curve is reduced to first degree equation (i.e. $x + y - 1 = 0$).

That is, the coeffs. of x^3(or y^3) and x^2(or y^2) become zero.

Therefore the asymptotes of the curve are

$$x - y = 0, \quad x + y + 1 = 0 \quad \text{and} \quad 2x + y + 1 = 0.$$

Method II. Let the equation of the curve be $f(x, y) = 0$.

In the given equation of the curve, we collect together the terms of the highest powers in x and y and find out their real linear factors. In general, we shall get one asymptote parallel to each factor $ax + by$. For, by putting $ax + by = 0$, the degree of the equation $f(x, y) = 0$ is reduced by one, therefore $ax + by = 0$ will cut the curve $f(x, y) = 0$ at one point at infinity. Therefore it is obvious that the st. line which is an asymptote of $f(x, y) = 0$ is parallel to $ax + by = 0$. We suppose that the equation of the asymptote is $ax + by + k = 0$. Now we have to choose the value of k in such a manner that this line will cut the curve $f(x, y) = 0$ at the second point at infinity. For this, we put the value of x in terms of y and k (e.g. $\because ax + by + k = 0$ $\therefore x = -\dfrac{by + k}{a}$) or the value of y in terms of x and k in $f(x, y) = 0$.

Thus, in the equation that will be obtained, we shall not get the highest degree term but the equation will begin with the term whose degree is 1 less than the previous one. Now we equate to zero the coeff. of the highest power of x (or y) in the resulting equation. From this, the value of k is obtained. Similarly we shall continue the process corresponding to each linear factor so that we get the equation of the asymptote parallel to each linear factor.

EXAMPLE : *Find the asymptotes of* $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0.$

SOLUTION : This curve is of 3rd degree.

Here the terms of the highest degree in x and y i.e. terms of 3rd degree are

$$(x^2 - y^2)(x + 2y) = 0$$

i.e. $$(x - y)(x + y)(x + 2y) = 0.$$

Therefore the curve has three asymptotes:

1st parallel to $x - y = 0$,

the 2nd parallel to $x + y = 0$,

and the 3rd parallel to $x + 2y = 0$.

Let $x - y = k_1$ be parallel to $x - y = 0$ so that $y = x - k_1$.

Now putting $y = x - k_1$ in the given equation

$$(x - y)(x + y)(x + 2y + 1) + x + y + 1 = 0,$$

we get $$k_1(2x - k_1)(x + 2x - 2k_1 + 1) + x + x - k_1 + 1 = 0$$

$\Rightarrow$ $$k_1(2x - k_1)(3x - 2k_1 + 1) + 2x - k_1 + 1 = 0.$$

Equating to zero the term of the highest degree in x

i.e. the coeff of x^2, we get $6k_1 = 0$ i.e. $k_1 = 0$..

$\therefore$ The equation of one asymptote is $x - y = 0$.

Similarly by taking the equation of another asymptote, parallel to $x + y = 0$, as $x + y = k_2$, it can be proved that $k_2 = 0$.

$\therefore$ The equation of another asymptote will be $x + y = 0$.

Similarly the equation of the third asymptote will be $x + 2y + 1 = 0$.

Hence the equations of three asymptotes are

$$x - y = 0, \quad x + y = 0 \quad \text{and} \quad x + 2y + 1 = 0.$$

Note : From method I also, the asymptotes of the curve will be

$$x - y = 0, \quad x + y = 0, \; x + 2y + 1 = 0.$$

Method III. Let the equation of the curve be put in the form

$$(ax + by + c)P_{n-1} + Q_{n-1} = 0 \qquad \ldots(1)$$

where P_{n-1} and Q_{n-1} are functions of x and y in which no one is of degree greater than $(n-1)$ and there is at least one term in P_{n-1} which is of $(n-1)$th degree. It has been assumed like this so that the equation of the curve may be of nth degree. If $ax+by+c=0$, then the equation of the curve becomes one degree less. Therefore the equation of the asymptote will be parallel to $ax+by+c=0$.

Let us suppose that the equation is $ax+by-k=0$.

Then expressing the given equation in terms of x, we get

$$(c+k)P_{n-1}' + Q_{n-1}' = 0 \qquad ...(2)$$

where y in P_{n-1} and Q_{n-1} has been expressed in terms of x.

Now, according to the last method, we shall get the value of k by equating to zero, the coeff. of x^{n-1} in (2). By equating to zero the coeff. of x^{n-1} will mean that the one root x of (2) will be $x=\infty$. Therefore

$$c+k = \underset{x\to\infty}{Lt} -\frac{Q_{n-1}'}{P_{n-1}'} \qquad ...(3)$$

Now, we leave both x and y unaltered in the expression on the L.H.S., then from the equation.

$$ax+by=k, \text{ we have } by = k-ax$$

i.e. $$b\frac{y}{x} = \frac{k}{x} - a.$$

$\therefore$ when $x\to\infty$, then $Lt\,\frac{y}{x} = -\frac{a}{b}$

i.e. $$Lt\,y = -\frac{a}{b}\,Lt\,x$$

$\because$ when $x\to\infty$ then $y\to\infty$, be the direction of $by+ax=0$.

$\therefore$ From (3), $c+k = \underset{\substack{x\to\infty\\ y\to\infty}}{Lt} -\frac{Q_{n-1}}{P_{n-1}}$; in the direction of $ax+by=0$

$\Rightarrow$ $$k = -c - \underset{\substack{x\to\infty\\ y\to\infty}}{Lt}\frac{Q_{n-1}}{P_{n-1}}$$; in the direction of $ax+by=0$.

$\therefore$ The equation of the asymptote is

$$ax+by+c = -\underset{\substack{x\to\infty\\ y\to\infty}}{Lt}\frac{Q_{n-1}}{P_{n-1}}$$; in the direction of $ax+by=0$

which is obtained from the equation (1).

In the practical way, we find out the limit as follows :

Since $ax + by = 0$, therefore taking $x = \frac{1}{t}$, we have

$$a \cdot \frac{1}{t} + by = 0 \Rightarrow by = -\frac{a}{t} \quad \therefore \quad y = -\frac{a}{bt}.$$

$\therefore$ when $x \to \infty$, $y \to \infty$, then $t \to 0$. At last putting $x = \frac{1}{t}$ and $y = -\frac{a}{bt}$ and putting $t = 0$ in the R.H.S. we get the limit.

EXAMPLE : *Find the asymptotes of the curve* $(x + y)(x^4 + y^4) = a(x^4 + a^4)$.

SOLUTION : The equation of the curve is of fifth degree and this can be written as follows :

$$(x + y)P_4 + Q_4 = 0.$$

$\therefore$ asymptote will be parallel to $x + y = 0$.

Hence the equation of the asymptote will be

$$x + y = \underset{y \to \infty}{\underset{x \to \infty}{Lt}} \frac{a(x^4 + a^4)}{x^4 + y^4}; \text{ in the direction of } x + y = 0.$$

Let $x = \frac{1}{t} \quad \therefore \quad y = -x = -\frac{1}{t}$.

Then $x + y = a \underset{t \to 0}{Lt} \left(\frac{\frac{1}{t^4} + a^4}{\frac{1}{t^4} + \frac{1}{t^4}} \right)$

$$= a \underset{t \to 0}{Lt} \left(\frac{1 + a^4 t^4}{2} \right) = a \cdot \frac{1}{2} = \frac{a}{2}.$$

Hence the equation of one asymptote is

$$x + y = \frac{a}{2}.$$

Method IV. Let the equation of the curve be of the form

$$(ax + by + c)P_{n-1} + Q_{n-2} = 0. \qquad \ldots(4)$$

It is obvious that $ax + by + c = 0$ is one asymptote of the curve (4) because if we put $ax + by + c = 0$ in (4), then the terms of nth degree and $(n - 1)$th degree become absent from the curve.

This means that the st. line $ax + by + c = 0$ cuts the curve at two points at infinity.

But if $ax + by + c$ is one of the factors of P_{n-1}, then the equation (4) will be of the following form :

$$(ax + by + c)^2 P_{n-2} + Q_{n-2} = 0.$$

In this case, if we put $ax + by + c = 0$, then as before, it is evident that the line $ax + by + c = 0$ cuts the curve at two points at infinity. Thus if any line be drawn parallel to $ax + by + c = 0$, then that line will cut the curve at two points at infinity. In this case, point at infinity is a 'double point'; i.e. the curve has two branches at a point at infinity. But of all these parallel lines, there are only two st. lines which cut the curve at three points at infinity and their equations will be obtained by taking the limit of

$$(ax + by + c)^2 = -\frac{Q_{n-2}}{P_{n-2}}$$

when $x \to \infty$, $y \to \infty$, in the direction of $ax + by = 0$.

Hence in this case, the equation of the reqd. asymptotes will be

$$ax + by + c = \pm \lim_{\substack{x \to \infty \\ y \to \infty}} \sqrt{-\frac{Q_{n-2}}{P_{n-2}}} \qquad \ldots(5)$$

in the direction of $ax + by = 0$

To find out the limit of R.H.S., we put $x = \frac{1}{t}$ in the equation $ax + by = 0$, then we find $y = -\frac{a}{bt}$.

$\therefore$ We put $x = \frac{1}{t}$ and $y = -\frac{a}{bt}$ in the expression on the R.H.S. and then we take the limit as $t \to 0$.

EXAMPLE : *Find the asymptotes of the curve* $(x + y)^2(x + 2y + 2) = x + 9y + 2$.

(P.U. 44A, 48A, 49S, 51A, 64, 70A, 88, 95H; B.U. 54A; M.U. 64S, 67A, 75A, 79A; Bhag. 68A, 86; R.U. 71A)

SOLUTION : If we put $x + 2y + 2 = 0$ in the given equation, then the equation of the curve becomes of the first degree $x + 9y + 2 = 0$ i.e. the terms containing the highest power 3 and the next highest power 2 becomes zero.

Therefore $x + 2y + 2 = 0$ is the equation of an asymptote. We find out the remaining two asymptotes as follows:

The given curve is of this form :

$$(x+y)^2 P_{n-1} + F_{n-1} = 0$$

i.e. here $(x+y)^2 = \dfrac{x+9y+2}{x+2y+2}$.

$\therefore$ The equations of other two asymptotes will be

$$x+y = \pm \underset{\substack{x+y=0 \\ x\to\infty,\ y\to\infty}}{Lt} \sqrt{\frac{x+9y+2}{x+2y+2}}$$

Now we put $x = \dfrac{1}{t}$ and $y = -\dfrac{1}{t}$ so that $t \to 0$.

$$\therefore \quad x+y = \pm \underset{t\to 0}{Lt} \sqrt{\frac{\frac{1}{t} - \frac{9}{t} + 2}{\frac{1}{t} - \frac{2}{t} + 2}}$$

$$= \pm \underset{t\to 0}{Lt} \sqrt{\frac{-8+2t}{-1+2t}} = \pm\sqrt{8} = \pm 2\sqrt{2}.$$

Hence the equations of three asymptotes will be

$$x + 2y + 2 = 0, \quad x + y = \pm 2\sqrt{2}.$$

Method V. If the equation of the curve is of the form

$$(ax+by)^2 P_{n-2} + (ax+by) Q_{n-2} + F_{n-2} = 0,$$

then putting $ax + by = 0$, the degree of the equation of the curve is reduced by two. Hence any line which is parallel to $ax + by = 0$, cuts the curve at two points at infinity. But of these, there are only two st. lines which cut the curve at three points at infinity. Thus we get two parallel asymptotes whose equations are given below :

$$(ax+by)^2 + (ax+by)\, Lt \frac{Q_{n-2}}{P_{n-2}} + Lt \frac{F_{n-2}}{P_{n-2}} = 0$$

where $x \to \infty$, $y \to \infty$; in the direction of $ax + by = 0$.

As before, putting $x = \dfrac{1}{t}$ and $y = -\dfrac{a}{bt}$ from $by + ax = 0$,

we find the limit for $t = 0$.

Thus we get the following equations of two parallel asymptotes:

$$(ax+by)^2 + (ax+by)l + m = 0.$$

Solving this, we get the equation of the asymptotes in the following form :

$$ax + by = d_1 \text{ and } ax + by = d_2.$$

EXAMPLE : *Find the asymptotes of the curve*

$$(y-x)^2(y+x) - 6y(y-x) + 4x = 0.$$

SOLUTION : Here the equation of the curve is of 3rd degree. We write the equation of the curve in the following way:

$$(y-x)^2 - \frac{6y}{y+x}\cdot(y-x) + \frac{4}{y+x}\cdot x = 0.$$

Hence the equation of two asymptotes will be

$$(y-x) - 6(y-x)\ Lt\frac{y}{y+x} + 4\ Lt\frac{x}{y+x} = 0 \qquad \ldots(1)$$

when $x \to \infty,\ y \to \infty$;

we put $x = \frac{1}{t}$ in the direction of $y - x = 0$.

Then $y = \frac{1}{t}$; from $y - x = 0$.

$\therefore$ From (1), $(y-x)^2 - 6(y-x)\ \underset{t\to 0}{Lt}\ \frac{\frac{1}{t}}{\frac{2}{t}} + 4\ \underset{t\to 0}{Lt}\ \frac{\frac{1}{t}}{\frac{2}{t}} = 0$

$\Rightarrow \quad (y-x)^2 - 6(y-x)\cdot\frac{1}{2} + 4\cdot\frac{1}{2} = 0$

$\Rightarrow \quad (y-x)^2 - 3(y-x) + 2 = 0$

$\Rightarrow \quad (y-x-1)(y-x-2) = 0$

$\Rightarrow \quad y - x - 1 = 0 \quad$ or $\quad y - x - 2 = 0.$

$\therefore$ The equations of two asymptotes are

$$y - x - 1 = 0,\ y - x - 2 = 0.$$

Again, if we put $y + x = 0$ in the given equation, then the equation of the curve becomes one degree less.

Therefore the other asymptotes will be parallel to $y + x = 0$.

From the equation of the curve,

$$y + x - \frac{6y}{y-x} + \frac{4x}{(y-x)^2} = 0.$$

$\therefore$ The reqd. asymptote will be

$$y + x = 6\ Lt\frac{y}{y-x} - 4\ Lt\frac{x}{(y-x)^2}$$

when $x \to \infty,\ y \to \infty$; in the direction of $y + x = 0$.

$\therefore$ Putting $x = \frac{1}{t}$ and $y = -\frac{1}{t}$,

$$y + x = 6\ Lt \frac{-\frac{1}{t}}{-\frac{1}{t} - \frac{1}{t}} - 4\ Lt \frac{\frac{1}{t}}{\left(-\frac{1}{t} - \frac{1}{t}\right)^2}$$

$$= 6 \cdot \frac{1}{2} + 0 = 3.$$

Therefore the equations of three asymptotes are

$$y - x - 1 = 0,\ y - x - 2 = 0,\ y + x = 3.$$

7.10. Correspondence between two definitions of Art. 7.1 & 7.2.

We have shown above that if the equation of the curve be $(ax + by + c)P_{n-1} + Q_{n-2} = 0$, then generally $ax + by + c = 0$ will be the equation of the asymptote.

Let $P(x, y)$ be any point on the curve, then the length of the perp. from $P(x, y)$ on the line $ax + by + c = 0$

$$p = \frac{ax + by + c}{\sqrt{a^2 + b^2}} = -\frac{1}{\sqrt{a^2 + b^2}} \cdot \frac{Q_{n-2}}{P_{n-1}};$$

from the equation of the curve.

Now we take $x \to \infty$, $y \to \infty$ in the direction of $ax + by = 0$.

Let $ax + by$ be a factor of P_{n-1}.

Since the power of denominator is greater than the power of numerator, therefore

$$\underset{\substack{x \to \infty \\ y \to \infty}}{Lt} \frac{Q_{n-2}}{P_{n-1}} = 0.$$

Hence $P \to 0$ when $x, y \to \infty$.

Thus the distance between the curve and the asymptote becomes less and less as $x, y \to \infty$.

Hence the second definition of the asymptote namely, 'an asymptote is that st. line such that its distance from a point on the curve decreases when the point goes to infinite distance away from the origin along the curve', is satisfied.

Example 31

Find the asymptotes of the following :

Method I + II

1. $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$ **(P.U. 61S; M.U. 71A, 75A; Bhag. 87)**
2. $(x - y)(x + y)(x + 2y - 1) = 2x + 3y + 5$
3. $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + 5x - 6y + 7 = 0$ **(P.U. 62A; P.U. 73)**
4. $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ **(R.U. 64; P.U. 68)**

   [Hints: $x^3 - 6x^2y + 11xy^2 - 6y^3 = (x - y)(x - 2y)(x - 3y)$**]**
5. $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0.$ **(B.U. 76; Mithila 76; M.U. 82H)**

Method III

6. $(x - y)(x + y)(x + 3y) + x^2 - 2xy + 3y^2 = 0$
7. $(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy = 1$ **(P.U. 86; R.U. 85)**

Method IV

8. $(x + y)^2(2x + 3y + 2) = x + 2y + 6$
9. $(x - y)^2(x + 2y - 1) = 3x + 4y + 5$ **(Bhag. 63A; R.U. 66A; B.U: 63A, 69S)**
10. $(2x - 3y + 1)^2(x + y) = 8x - 2y + 9$
11. $(x^2 - y^2)^2 = 2(x^2 + y^2)$ **(P.U. 59; B.U. 62S; M.U. 73A)**

    [From the given equation, we have

    $(x + y)^2(x - y)^2 = 2(x^2 + y^2)$]

Method V

12. $(y - x)^2x - 3y(y - x) + 2x = 0$ **(P.U. 93H)**
13. $x(x - y)^2 - 3(x^2 - y^2) + 8y = 0$ **(P.U. 59S; R.U. 67S, 83)**
14. $y(x - y)^3 = y(x - y) + 2$ **(P.U. 35A, 48S; M.U. 85)**
15. $x^2(x + y)(x - y)^2 + ax^3(x - y) + a^2y^3 = 0$
16. $(x - 2y)^2(x - y) - 4y(x - 2y) - (8x + 7y) = 0$
17. $(x + y)^2(x^2 - 4y^2) - (x + y)(2x^2 + 3xy + 7y^2) = 8.$ **(P.U. 38A)**
18. Find the parallel asymptotes of

    $(y - 2x)^2(y - x) + (y + 3x)(y - 2x) + 2x + 2y - 1 = 0$ **(B.U. 64A)**

ANSWERS

1. $y-x=0, y+x=0, x+2y+1=0$. 2. $x-y=0, x+y=0, x+2y-1=0$.
3. $x-y+2=0, 2x-3y+4=0, 4x-5y+6=0$.
4. $x-y=0, x-2y=0, x-3y=0$. 5. $x-y=0, 2x-y=0, 3x-y=0$.
6. $x-y+\frac{1}{4}=0, x+y-\frac{3}{2}=0, x+3y+\frac{9}{4}=0$.
7. $y=x, y=2x, y+x+1=0, y+2x+1=0$.
8. $2x+3y+2=0, x+y-1=0, x+y+1=0$.
9. $x+2y-1=0, x-y=\pm\sqrt{7/3}$
10. $x+y=0, 2x-3y-1=0, 2x-3y+3=0$.
11. $y+x=1, y+x=-1, y-x=1, y-x=-1$.
12. $y=x+2, y=x+1, x=3$. 13. $x+3=0, x-y=4, x-y=2$.
14. $y=0, x-y=0, x-y-1=0, x-y+1=0$.
15. $x=\pm a, 2x+2y+a=0, x-y-a=0, x-y+a=0$.
16. $x-y+4=0, x-2y=2\pm3\sqrt{3}$.
17. $x+y=0, x+y+2=0, 4x-8y=7, 4x+8y=9$.
18. $y-2x+3=0, y-2x+2=0$.

7.11. Intersection of a curve with its asymptotes

We known from Art 7.3 that a st. line cuts a curve of nth degree in n points, real or imaginary. But if the st. line is an asymptote of the curve, then its two points of intersection will be at infinity. Thus any asymptote of a curve of n th degree cuts the curve in $(n-2)$ points.

Hence the n asymptotes of a curve of nth degree cut it in $n(n-2)$ points.

Note : The equations of the asymptotes and of the curve may be respectively written as $P_n = 0$ and $P_n + Q_{n-2} = 0$ where Q_{n-2} is of degre$(n-2)$ at the most and P_n consists of n non-repeated linear factors. The n asymptotes, therefore intersect the curve again at points lying on the curve $Q_{n-2} = 0$.

Hence the $n(n-2)$ *points lie on a certain curve* $Q_{n-2} = 0$ *i.e. a curve of* $(n-2)$*th degree.*

Cor. (i) For a cubic, $n = 3$ and therefore the asymptotes cut the curve in $3(3-2) = 3$ points which lie on a curve of degree $(3-2) = 1$.
i.e. on a st. line.

(ii) For a quadric, $n = 4$ and therefore the asymptotes cut the curve in $4(4-2) = 8$ points which lie on a curve of degree $(4-2) = 2$
i.e. on a conic.

Miscellaneous Examples

EXAMPLE 1. *Find the asymptotes of the curve* $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ *and then prove that the asymptotes form a triangle of area* a^2.

(P.U. 60S; B.U. 55S; M.U. 66A, 77; Bhag. 64A)

SOLUTION : Simplifying the equation of the curve, we get

$$x^2y - y^3 - 2ay^2 + 5x - 7 = 0.$$

Here, equating the coeff. y of x^2 the highest power term of x to zero, we get the equation of the asymptote parallel to x-axis viz $y = 0$.

Again, putting $x = 1$, $y = m$, in the highest power i.e. in the 3rd degree term in the equation of the curve and putting that equal to zero, we get

$$\phi_3(m) = m - m^3 = 0$$

$$\Rightarrow \quad m(1 - m^2) = 0 \quad \Rightarrow \quad m(1 - m)(1 + m) = 0$$

$$\therefore \quad m = 0,\ m = 1,\ m = -1.$$

Again, $\phi_2(m) = 2 - am^2$.

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-2am^2}{1 - 3m^2} = \frac{2am^2}{1 - 3m^2}.$$

$\therefore$ when $m = 0$, then $c = 0$,

when $m = 1$, then $c = \dfrac{2a}{1 - 3} = -a$,

when $m = -1$, then $c = \dfrac{2a}{1 - 3} = -a$.

Putting the values of m and c, the equations of three asymptotes will be

$$y = 0.x + 0 \quad \text{i.e.} \quad y = 0 \qquad ...(1)$$

$$y = 1.x - a \quad \text{i.e.} \quad y = x - a \qquad ...(2)$$

$$y = -1.x - a \quad \text{i.e.} \quad y = -x - a \qquad ...(3)$$

Now we are required to find out that the area of the triangle enclosed by the asymptotes (1), (2) and (3) $= a^2$.

For this, we first of all, find out the co-ordinates of the three vertices of the triangle.

From (1) and (2), $x = a$, $y = 0$

From (1) and (3), $x = -a$, $y = 0$

and from (2) and (3), $2y = -2a \ \therefore \ y = -a \ \therefore \ x = 0$.

Miscellaneous Examples

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Again, putting $x = 1$, $y = m$, in the highest power i.e. in the 3rd degree term in the equation of the curve and putting that equal to zero, we get

$$\phi_3(m) = m - m^3 = 0$$

$$\Rightarrow \quad m(1 - m^2) = 0 \quad \Rightarrow \quad m(1 - m)(1 + m) = 0$$

$$\therefore \quad m = 0,\ m = 1,\ m = -1.$$

Again, $\phi_2(m) = 2 - am^2$.

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Now we are required to find out that the area of the triangle enclosed by the asymptotes (1), (2) and (3) $= a^2$.

For this, we first of all, find out the co-ordinates of the three vertices of the triangle.

From (1) and (2), $x = a$, $y = 0$

From (1) and (3), $x = -a$, $y = 0$

and from (2) and (3), $2y = -2a \ \therefore \ y = -a \ \therefore \ x = 0$.

Hence the co-ordinates of the three vertices of the triangle are

$$(a, 0), (-a, 0) \text{ and } (0, -a).$$

$\therefore$ Area of the triangle

$$= \frac{1}{2}[a(0 + a) + (-a)(-a - 0) + 0(0 - 0)]$$

$$= \frac{1}{2}(a^2 + a^2) = \frac{1}{2} \cdot 2a^2 = a^2.$$

Hence the result.

EXAMPLE 2. *Find the asymptotes of the cubic* $x^2y - xy^2 + xy + y^2 + x - y = 0$ *and show that they cut the curve again in three points lying on a straight line.*

(P.U. 57A, 64S, 69A, 73A; B.U. 61S, 63A, 65A; Mithila 80)

SOLUTION : Here putting $x = 1$, $y = m$

$$\phi_3(m) = m - m^2 = 0$$

$$\Rightarrow \quad m(m - 1) = 0 \quad \therefore \quad m = 0, 1.$$

Again $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{m + m^2}{1 - 2m}$$

When $m = 0$, then $c = 0$.

When $m = 1$, then $c = -\frac{2}{-1} = 2$.

$\therefore$ The equations of two asymptotes are

$$y = 0.x + 0 \quad \text{i.e. } y = 0$$

$$y = 1.x + 2 \quad \text{i.e. } y = x + 2.$$

Again, equating to zero the higher powers in x and y, the equations of the asymptotes parallel to co-ordinate axes are

$$y = 0 \text{ and } 1 - x = 0 \text{ i.e. } x = 1.$$

$\therefore$ The equations of three asymptotes are

$$y = 0, (x - 1) = 0, (x - y + 2) = 0.$$

Now the joint equation of the three asymptotes will be

$$y(x - 1)(x - y + 2) = 0$$

$$\Rightarrow \quad x^2y - xy^2 + xy + y^2 - 2y = 0$$

which we write as $F_3 = 0$. ...(1)

But the equation of the curve is

$$x^2y - xy^2 + xy + y^2 - 2y + (x + y) = 0$$

i.e.

$$F_3 + (x + y) = 0.$$

Hence the point of intersection of the curve and asymptotes satisfy $x + y = 0$ which represents a st. line.

But we know that a st. line cuts a curve of 3rd degree in 3 points. But each asymptote passes through two points at infinity. Therefore it will cut the given curve at (3 – 2) or 1 point more. Since the number of asymptotes = 3, therefore the number of points of intersection of the curve and asymptotes = 3 × 1 = 3.

Thus all the three points of intersection of the curve and asymptotes lie on a st. line. Hence the result.

EXAMPLE 3. *Determine the asymptotes of the curve*

$$4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0.$$

and show that they pass through the points of intersection of the curve with the ellipse $x^2 + 4y^2 = 4$. **(P.U. 54H, 59H; B.U. 62H, 64H, 66H; Bhag. 70H)**

SOLUTION : The equation of the curve can be written as

$$(4x^4 + 4y^4 - 17x^2y^2) - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

$$\Rightarrow \quad (x^2 - 4y^2)(4x^2 - y^2) + 4x(x^2 - 4y^2) + 2(x^2 - 2) = 0$$

$$\Rightarrow \quad (x^2 - 4y^2)(4x^2 - y^2 + 4x) + 2(x^2 - 2) = 0$$

$$\Rightarrow \quad (x^2 - 4y^2)\{(4x^2 + 4x + 1) - y^2 - 1\} + 2(x^2 - 2) = 0$$

$$\Rightarrow \quad (x^2 - 4y^2)\{(2x + 1)^2 - y^2\} - (x^2 - 4y^2)\ (2x^2 - 4) = 0$$

$$\Rightarrow \quad (x - 2y)(x + 2y)(2x + 1 + y)(2x + 1 - y) + \{x^2 + 4y^2 - 4\} = 0$$

Hence the required asymptotes are

$$x - 2y = 0,\ \ x + 2y = 0,\ \ 2x + y + 1 = 0,\ \ 2x - y + 1 = 0. \qquad \ldots(1)$$

From the equation of the curve and the equation (1), it is evident that the points of intersection of the curve and the ellipse $x^2 + 4y^2 = 4$ satisfy the equations of the asymptotes.

Hence the result.

EXAMPLE 4. *If from the origin a right line be drawn parallel to any of the asymptotes of the cubic* $y(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) = x^3$, *show that the portion of this line intercepted between the origin and the line* $gx + fy + c = 0$ *is bisected by the curve.*

(P.U. 53H, 60H, 62H; B.U. 62H; M.U. 64H)

SOLUTION : Let $y = mx + n$ be one of the asymptotes.

Substituting $mx + n$ for y in the equation of the curve, we get

$$(mx + n)\{ax^2 + 2hx(mx + n) + b(mx + n)^2 + 2gx + 2f(mx + n) + c\} - x^3 = 0$$

If $y = mx + n$ is an asymptote of the given curve, then we must have coeff. of $x^3 = 0$ (also coeff. of $x^2 = 0$).

Hence the coeff. of x^3 equated to zero gives

$$ma + 2hm^2 + bm^3 - 1 = 0 \qquad ...(1)$$

Now, the line though the origin parallel to $y = mx + n = 0$ is $y = mx$.

Let the line $y = mx$ meet the given line $gx + fy + c = 0$ at the point P.

$$\therefore\ gx + fmx + c = 0 \Rightarrow x(g + fm) = -c \Rightarrow x = \frac{-c}{g + fm}.$$

$$\therefore\ y = mx = \frac{-mc}{g + fm}.$$

That is, the co-ordinates of P are $\left(\frac{-c}{g + fm}, \frac{-mc}{g + fm}\right)$.

Let M be the middle point of OP. Then the co-ordinates of M are

$$\left\{\frac{-c}{2(g + fm)}, \frac{-mc}{2(g + fm)}\right\}.$$

If M lies on the given curve, then

$$-\frac{mc}{2(g + fm)}\left[a \cdot \frac{c^2}{4(g + fm)^2} + 2h\left(\frac{-c}{2(g + fm)}\right)\left(\frac{-mc}{2(g + fm)}\right) + b \cdot \frac{m^2c^2}{4(g + fm)^2}\right.$$

$$\left. + 2g\left(\frac{-c}{2(g + fm)}\right) + 2f\left(\frac{-mc}{2(g + fm)}\right) + c\right] - \left(\frac{-c}{2(g + fm)}\right)^3 = 0$$

$$\Rightarrow -\frac{mc}{2(g + fm)}\left[\frac{ac^2}{4(g + fm)^2} + \frac{mc^2h}{2(g + fm)^2} + \frac{bm^2c^2}{4(g + fm)^2} - \frac{gc}{g + fm}\right.$$

$$\left. - \frac{fmc}{g + fm} + c\right] + \frac{c^3}{8(g + fm)^3} = 0$$

$$\Rightarrow \frac{-mc}{2(g + fm)}\left[\frac{ac^2 + 2mc^2h + bm^2c^2}{4(g + fm)^2} - \frac{gc + fmc}{g + fm} + c\right] + \frac{c^3}{8(g + fm)^3} = 0$$

$$\Rightarrow \frac{-mc}{2(g + fm)}\left[\frac{c^2(a + 2hm + bm^2)}{4(g + fm)^2} - \frac{c(g + fm)}{g + fm} + c\right] + \frac{c^3}{8(g + fm)^3} = 0$$

$$\Rightarrow \frac{-c^3}{8(g + fm)^3}[m(a + 2hm + bm^2) - 1] = 0$$

$$\Rightarrow am + 2hm^2 + bm^3 - 1 = 0$$

which is true because of (1).

Hence the curve passes through M.

EXAMPLE 5. *Prove that a cubic curve which crosses itself at some point cannot have parallel asymptotes.*

(B.U. 56H, 60H; P.U. 57H, 61H; R.U. 62H; Bhag. 62H; M.U. 65H)

SOLUTION : Let the point , where the cubic curve crosses itself be taken as the origin.

If possible, let the curve possess two of its asymptotes which are parallel to each other.

Let the axes be so chosen that $x + m = 0$ and $x + n = 0$ be two parallel asymptotes and let the third asymptote be $ax + by + c = 0$.

Then the equation of the cubic curve is of the form

$$(x + m)(x + n)(ax + by + c) - (px + qy + r) = 0$$

$$\Rightarrow \quad \{x(x + m + n) + mn\}.(ax + by + c) - (px + qy + r) = 0 \quad ...(1)$$

Since the curve has two tangents at the origin therefore the constant term and the first degree terms must vanish.

$\therefore$ const. term viz. $mnc - r = 0$;

coeff. of x viz. $c(m + n) + amn - p = 0$;

coeff. of y viz. $bmn - q = 0$.

Thus $r = mnc$, $p = c(m + n) + amn$, and $q = bmn$.

Hence (1) becomes

$$\{x(x + m + n) + mn\}(ax + by + c) + [\{amn + c(m + n)\}x + bmny + mnc] = 0$$

$$\Rightarrow \quad x\{(x + m + n)(ax + by + c) - c(m + n)\} = 0$$

Hence the cubic degenerates into a line and a conic.

If $c = 0$ or $m + n = 0$, then the cubic degenerates into three st. lines.

Hence a cubic curve which crosses at some point cannot have parallel asymptotes.

EXAMPLE 6. *Find the condition that the three asymptotes of the curve*

$$a_0 + 3b_0x + 3b_1y + 3c_0x^2 + 6c_1xy + 3c_2y^2 + d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3 = 0$$

should pass through a common point.

(P.U. 51H, 56H; B.U. 53H, 56H; Bhag. 66H)

SOLUTION : Let the common point be (α, β).

Shift the origin to (α, β).

Then the equation of the curve becomes

$$a_0 + 3b_0(x+\alpha) + 3b_1(y+\beta) + 3c_0(x+\alpha)^2 + 6c_1(x+\alpha)(y+\beta)$$
$$+ 3c_2(y+\beta)^2 + d_0(x+\alpha)^3 + 3d_1(x+\alpha)^2(y+\beta)$$
$$+ 3d_2(x+\alpha)(y+\beta)^2 + d_3(y+\beta)^3 = 0$$

$$\Rightarrow\ a_0 + 3b_0(x+\alpha) + 3b_1(y+\beta) + 3c_0(x^2 + 2\alpha x + \alpha^2)$$
$$+ 6c_1(xy + x\beta + \alpha y + \alpha\beta) + 3c_2(y^2 + 2\beta y + \beta^2) + d_0(x^3 + 3x^2\alpha$$
$$+ 3x\alpha^2 + \alpha^3) + 3d_1(x^2 + 2\alpha x + \alpha^2)(y+\beta) + 3d_2(x+\alpha)(y^2 + 2\beta y + \beta^2)$$
$$+ d_3(y^3 + 3y^2\beta + 3y\beta^2 + b^3) = 0$$

Since the asymptotes of this curve pass through the origin, therefore the second degree terms must vanish.

Here $3c_0x^2 + 6c_1xy + 3c_2y^2 + 3d_0x^2\alpha + 3d_1x^2\beta + 6d_1\alpha xy + 6d_2\beta xy$
$$+3d_2\alpha y^2 + 3d_3\beta y^2 = 0$$

$$\Rightarrow\ (3c_0 + 3d_0\alpha + 3d_1\beta)x^2 + (6c_1 + 6d_1\alpha + 6d_2\beta)xy$$
$$+ (3c_2 + 3d_2\alpha + 3d_3\beta)y^2 = 0$$

$$\Rightarrow\ 3c_0 + 3d_0\alpha + 3d_1\beta = 0$$
$$6c_1 + 6d_1\alpha + 6d_2\beta = 0$$
$$3c_2 + 3d_2\alpha + 3d_3\beta = 0$$

Eliminating (α, β), we get $\begin{vmatrix} c_0 & d_0 & d_1 \\ c_1 & d_1 & d_2 \\ c_2 & d_2 & d_3 \end{vmatrix} = 0.$

Example 32

1. Find the equation of the cubic which has the same asymptotes as those of the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points (0, 0),(1, 0) and (0, 1). **(P.U. 68A)**

[Hint : The given eqn. is $(x - y)(x - 2y)(x - 3y) + x + y + 1 = 0$.

$\therefore$ The asymptotes are $x - y = 0$, $x - 2y = 0$, $x - 3y = 0$.

The general eqn. of the cubic which has the same asymptotes as those of the given curve is of the form

$$(x - y)(x - 2y)(x - 3y) + ax + by + c = 0.$$

By putting the co-ordinates of three points we determine a, b, c.]

2. Show that the asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.

3. Show that the asymptotes of the curve $(x^2 - y^2)^2 = 8(x^2 + y^2)$ form a square. **(Bhag. 70A; M.U.73A)**

4. Show that the four asymptotes of the curve

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$$ form a square.

(R.U. 65S, 72A; Bhag. 66S; P.U. 72A)

5. Show that the asymptotes to the curve

$$x^2y^2 - x^2 - y^2 - x - y + 1 = 0$$

form a square through two of whose angular points the curve passes.

6. Show that the asymptotes of the curve

$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$

form a square two of whose angular points lie on the curve. **(P.U.53)**

[Hint : By equating to zero the coeffs. of highest powers of x and y we find that the asymptotes parallel to the axis of x and y are $y = \pm a$, $x = \pm a$.

Evidently the line $x = +a$, $x = -a$, $y = +a$, $y = -a$ form a square].

7. Show that the asymptotes of the cubic

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$

cut the curve again in three points which lie on the st. line $x - y + 1 = 0$.

8. Show that the four asymptotes of the curve

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle $x^2 + y^2 = 1$.

(P.U.86)

[The given eqn. is

$$(x - y)(y - 2x)(x + y + 1)(y + 2x + 1) + x^2 + y^2 - 1 = 0\,].$$

ANSWER

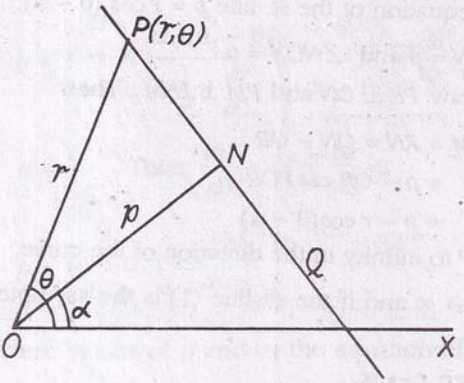
1. $x^3 - 6x^2y + 11xy^2 - 6y^3 - x = 0$

7.12. Asymptotes of Polar curves

First we derive the equation of a st. line in polar form.

Statement : *To prove that the polar equation of a st. line is* $p = r\cos(\theta - \alpha)$ *where* p *is the length of the perpendicular from the pole to the st. line and* α *is the angle which this st. line makes with the initial line.*

Let PQ be any line on which a perpendicular ON has been drawn from O (pole).



Let $ON = p$ and let ON make an angle α with OX. We desire to find the equation of the line PQ.

Let $P(r, \theta)$ be any point on the line. Join OP. Then $OP = r$ and $\angle POX = \theta$.

$\therefore \qquad \angle NOP = \theta - \alpha.$

Now from ΔPON, $ON = OP \cos PON$

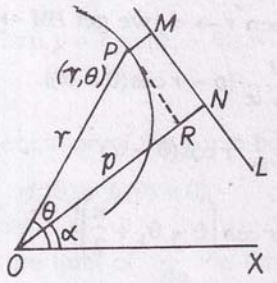
$\therefore \qquad p = r\cos(\theta - \alpha)$

which is the reqd. equation of the line.

7.13. Asymptotes of the polar curve $r = f(\theta)$.

Let LNM be the asymptote of the curve $r = f(\theta)$ whose equation is

$$p = r\cos(\theta - \alpha). \qquad ...(1)$$



Now we have to choose the values of p and α in such a manner that the distance between the curve and the st. line $\to 0$ when $r \to \infty$.

Let $P(r, \theta)$ be any point on the curve.

$\therefore \qquad OP = r$ and $\angle POX = \theta$.

From O, draw ON perpendicular on the line LNM.

Then from the equation of the st. line $p = r\cos(\theta - \alpha)$,

$$ON = p \text{ and } \angle NOX = \alpha .$$

Now from P, draw $PR \perp ON$ and $PM \perp LNM$. Then

$$\begin{aligned} PM = RN &= ON - OR \\ &= p - OP\cos POR \\ &= p - r\cos(\theta - \alpha) \end{aligned} \qquad \ldots(2)$$

Now we take P to infinity in the direction of the curve.

In this case, $r \to \infty$ and if the st. line (1) is the asymptote of the curve, then $PM \to 0$.

Let $\theta \to \theta_1$ when $r \to \infty$.

Now from (2),

$$\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha) .$$

Taking its limit when $r \to \infty$, then since

$$\frac{PM}{r} \to 0, \frac{p}{r} \to 0, \theta \to \theta_1$$

we find that $0 = 0 - \cos(\theta_1 - \alpha)$

$$\Rightarrow \quad \cos(\theta_1 - \alpha) = 0$$

$$\Rightarrow \quad \theta_1 - \alpha = \frac{\pi}{2} \qquad \therefore \quad \alpha = \theta_1 - \frac{\pi}{2} .$$

Again, from (1),

$$PM = p - r\cos(\theta - \alpha).$$

$\therefore$ Taking the limit when $r \to \infty$, we get $PM \to 0$.

$$\therefore \quad 0 = \underset{r \to \infty}{Lt} \{p - r\cos(\theta - \alpha)\}.$$

$$\Rightarrow \quad p = \underset{r \to \infty}{Lt} r\cos(\theta - \alpha)$$

$$= Lt\left\{r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right)\right\}$$

$$= \underset{\substack{r \to \infty \\ \theta \to \theta_1}}{Lt} r\sin(\theta_1 - \theta) .$$

$$= Lt \frac{\sin(\theta_1 - \theta)}{\frac{1}{r}}$$

which is of the form $\frac{0}{0}$.

$$\therefore \quad p = \underset{\theta \to \theta_1}{Lt_{r \to \infty}} \frac{\cos(\theta_1 - \theta)}{\frac{1}{r^2} \cdot \frac{dr}{d\theta}} = \underset{\theta \to \theta_1}{Lt_{r \to \infty}} r^2 \frac{d\theta}{dr}.$$

Let $u = \frac{1}{r}$. Then $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$\therefore \quad p = \underset{\theta \to \theta_1}{Lt_{u \to 0}} \left(-\frac{d\theta}{du}\right).$$

Thus, with these values of p and α, the equation of the asymptote becomes

$$p = r\cos(\theta - \alpha)$$

i.e. $$Lt\left(-\frac{d\theta}{du}\right) = r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) = r\sin(\theta_1 - \theta)$$

where $\theta \to \theta_1$ when $r \to \infty$ i.e. $u \to 0$.

7.14. Working rule for obtaining asymptotes to polar curves.

(i) Put $r = \frac{1}{u}$ in the given equation and find the limit of θ when $u \to 0$.

(ii) Let $\theta = \theta_1$ be any one limit among the various possible limits of θ.

(iii) After this, we shall find the limit of $\left(-\frac{d\theta}{du}\right)$ when $u \to 0$ and $\theta \to \theta_1$.

Let its limit be p. Then $p = r\sin(\theta_1 - \theta)$, will be the equation of asymptote corresponding to θ_1.

Note : If the polar equation of the curve be of the form

$$rf_1(\theta) + f_0(\theta) = 0,$$

then we can find the limit of $\frac{d\theta}{du}$ in this way also . From the given equation,

$$rf_1(\theta) = -f_0(\theta) \quad \therefore \quad r = -\frac{f_0(\theta)}{f_1(\theta)}$$

$$\Rightarrow \quad u = -\frac{f_1(\theta)}{f_0(\theta)}.$$

$\therefore$ When $u \to 0$, then $f_1(\theta) \to 0$. Let $\theta = \theta_1$ satisfies this equation so that $f_1(\theta_1) = 0$.

Now $$\frac{du}{d\theta} = -\frac{f_0(\theta)f_1'(\theta) - f_1(\theta)f_0'(\theta)}{\{f_0(\theta)\}^2}$$

$\therefore$ $$\frac{d\theta}{du} = -\frac{\{f_0(\theta)\}^2}{f_0(\theta)f_1'(\theta) - f_1(\theta)f_0'(\theta)}$$

$\therefore$ $$\underset{\theta \to \theta_1}{Lt}\left(-\frac{d\theta}{du}\right) = \frac{\{f_0(\theta_1)\}^2}{f_0(\theta_1)f_1'(\theta_1)}; \quad \because f_1(\theta_1) = 0$$

$$= \frac{f_0(\theta_1)}{f_1'(\theta_1)}.$$

Worked out Examples

EXAMPLE 1. *Find the asymptotes of the curve* $r\theta = a$. **(M.U. 72A)**

SOLUTION : Here $\theta = \frac{a}{r} = au \quad \because u = \frac{1}{r}$.

$\therefore$ when $u \to 0$, then $\theta \to 0$ i.e. $\theta_1 = 0$.

Again from $\theta = au$, $\frac{d\theta}{du} = a$.

$\therefore$ The equation of the asymptote is

$$Lt\left(-\frac{d\theta}{du}\right) = r\sin(\theta_1 - \theta)$$

$\Rightarrow$ $$-a = r\sin(0 - \theta) = -r\sin\theta$$

$\therefore$ $$r\sin\theta = a.$$

EXAMPLE 2. *Find the asymptotes of the curve* $r = \frac{2a}{1 - 2\cos\theta}$

SOLUTION : Here $u = \frac{1}{r} = \frac{1 - 2\cos\theta}{2a}$. ...(1)

$\therefore$ When $u \to 0$, then $1 - 2\cos\theta \to 0$ i.e. $\cos\theta \to \frac{1}{2}$.

$\therefore$ $$\theta = \pm\frac{\pi}{3}.$$

$\therefore$ We take $\theta_1 = \pm\frac{\pi}{3}$.

Now from (1),

$$\frac{du}{d\theta} = \frac{1}{2a}(0 + 2\sin\theta) = \frac{1}{a}\sin\theta.$$

$$\therefore \quad -\frac{d\theta}{du} = -\frac{a}{\sin\theta}.$$

Thus $\underset{\theta \to \frac{\pi}{3}}{Lt}\left(-\frac{d\theta}{du}\right) = -\underset{\theta \to \frac{\pi}{3}}{Lt}\frac{a}{\sin\theta} = -\frac{a}{\sin\frac{\pi}{3}}$

$$= -\frac{a.2}{\sqrt{3}} = -\frac{2a}{\sqrt{3}}$$

and $\underset{\theta \to -\frac{\pi}{3}}{Lt}\left(-\frac{d\theta}{du}\right) = -\underset{\theta \to -\frac{\pi}{3}}{Lt}\frac{a}{\sin\theta} = +\frac{a}{\sin\frac{\pi}{3}}$

$$= \frac{2a}{\sqrt{3}}.$$

Hence there are two equations of asymptotes.

(i) $$-\frac{2a}{\sqrt{3}} = r\sin\left(\frac{\pi}{3} - \theta\right)$$

$\Rightarrow$ $$-\frac{2a}{\sqrt{3}} = r\left(\sin\frac{\pi}{3}\cdot\cos\theta - \cos\frac{\pi}{3}\cdot\sin\theta\right)$$

$$= r\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$$

$$= \frac{r}{2}(\sqrt{3}\cos\theta - \sin\theta)$$

$\Rightarrow$ $$-4a = r(3\cos\theta - \sqrt{3}\sin\theta)$$

$\therefore$ $$r(\sqrt{3}\sin\theta - 3\cos\theta) = 4a.$$

(ii) $$\frac{2a}{\sqrt{3}} = r\sin\left(-\frac{\pi}{3} - \theta\right)$$

$$= -r\left(\sin\frac{\pi}{3}\cdot\cos\theta + \cos\frac{\pi}{3}\cdot\sin\theta\right)$$

$$= -r\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)$$

$$= -\frac{r}{2}(\sqrt{3}\cos\theta + \sin\theta)$$

$\Rightarrow$ $$-4a = r(3\cos\theta + \sqrt{3}\sin\theta)$$

$\therefore$ $$r(\sqrt{3}\sin\theta + 3\cos\theta) = -4a.$$

EXAMPLE 3. *Show that there is an infinite series of parallel asymptotes to the curve* $r = \dfrac{a}{\theta \sin\theta} + b$

and show that their distances from the pole are in H.P.

(P.U. 52H, 58H, 77H; B.U. 54H, 63H; M.U. 63H; Bhag. 67H)

SOLUTION : Here $r = \dfrac{a}{\theta \sin\theta} + b = \dfrac{a + b\theta \sin\theta}{\theta \sin\theta}$. ...(1)

From (1), $u = \dfrac{1}{r} = \dfrac{\theta \sin\theta}{a + b\theta \sin\theta}$. ...(2)

$\therefore$ when $u \to 0$, then $\theta \sin\theta \to 0$ ...(3)

i.e. $\theta = n\pi$, $n = 0, \pm 1, \pm 2, \pm 3...$

Now differentiating (2) w.r.t. θ, we get

$$\frac{du}{d\theta} = \frac{(a + b\theta \sin\theta)(\theta \cos\theta + \sin\theta) - b\theta \sin\theta(\theta \cos\theta + \sin\theta)}{(a + b\theta \sin\theta)^2}$$

$$= \frac{(\theta \cos\theta + \sin\theta)(a + b\theta \sin\theta - b\theta \sin\theta)}{(a + b\theta \sin\theta)^2}$$

$$= \frac{a(\theta \cos\theta + \sin\theta)}{(a + b\theta \sin\theta)^2}.$$

$$\therefore \quad -\frac{d\theta}{du} = -\frac{(a + b\theta \sin\theta)^2}{a(\theta \cos\theta + \sin\theta)}$$

$$\therefore \quad \underset{\theta \to n\pi}{Lt}\left(-\frac{d\theta}{du}\right) = -\frac{(a + bn\pi \sin n\pi)^2}{a(n\pi \cos n\pi + \sin n\pi)} = -\frac{a^2}{an\pi \cos n\pi}, \quad \sin n\pi = 0$$

$$= -\frac{a}{n\pi(-1)^n}.$$

$\therefore$ The equation of the asymptotes will be

$$r\sin(n\pi - \theta) = \frac{-a}{n\pi(-1)^n}$$

$$\Rightarrow \quad r\{\sin n\pi \cos\theta - \cos n\pi \sin\theta\} = \frac{-a}{n\pi(-1)^n}$$

$$\Rightarrow \quad r(-1)(-1)^n \sin\theta = \frac{-a}{n\pi(-1)^n}$$

$$\Rightarrow \quad -r(-1)^{2n}\sin\theta = \frac{-a}{n\pi}$$

$$\therefore \quad r\sin\theta = \frac{a}{n\pi}.$$ ...(4)

If $n = 0$, then R.H.S. of (4) becomes infinite. Therefore in this case we do not get any asymptote.

Thus corresponding to $n = \pm 1, \pm 2, \pm 3$ ..... we get infinite number of asymptotes.

The distance of those asymptotes from the pole will be always $r\sin\theta$ (If any radius vector makes an angle of θ with the initial line, then the length of the perpendicular from the origin on that line = $r\sin\theta$).

i.e. $\frac{a}{\pi}, \frac{a}{2\pi}, \frac{a}{3\pi}$ ...; on putting $n = 1, 2, 3, \ldots$

Obviously these distances are in H.P.

Example 33

1. Find the asymptotes of the following curves :

   (i) $r\theta = a$ (ii) $r\sin n\theta = a$

   (iii) $r\cos\theta = a\sin\theta$ **(B.U. 58H, 67H, 71H; M.U. 73H)**

   (iv) $r\sin\theta = 2\cos 2\theta$ (v) $r\cos 2\theta = a\sin 3\theta$

   (vi) $r = a\sec\theta + b\tan\theta$

2. Show that all the asymptotes of the curve $r\tan n\theta = a$ touch the circle $r = \frac{a}{n}$.

3. Show that the curve $r = a\sec m\theta + b\tan m\theta$ has two sets of asymptotes; members of the first set touching one fixed circle and those of other another fixed circle.

Answers

1. (i) $r\sin\theta = a$. (ii) $r\sin\left(\theta - \frac{m\pi}{n}\right) = \frac{a}{n}\sec m\pi$ where m is an integer.

   (iii) $r\cos\theta = \pm a$ (iv) $r\sin\theta = 2$.

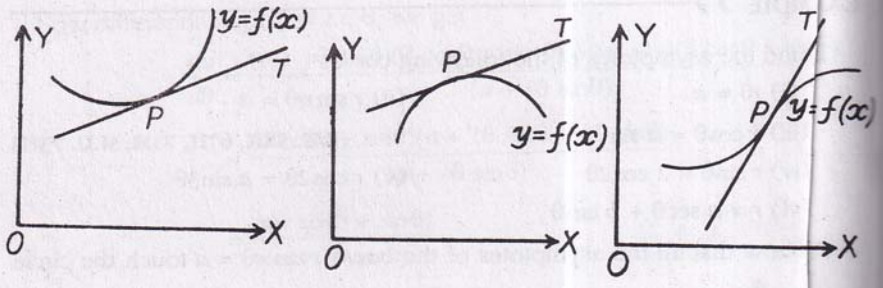
   (v) $a + 2r(\cos\theta + \sin\theta) = 0$; $a = 2r(\cos\theta - \sin\theta)$ (vi) $r\cos\theta \doteq a \pm b$.

8

CONCAVITY, CONVEXITY : POINTS OF INFLEXION

8.1. *Definition*—Let P be a point on the curve $y = f(x)$. Draw the tangent PT at P. We suppose that this tangent is not parallel to y-axis.

Now, there are three mutually exclusive possibilities regarding the nature of the curve vis-a-vis the tangent at P.



(i) A portion of the curve on both sides of P, however small it may be, lies above the tangent PT (i.e. towards the positive direction of y-axis. In this case, we say that the curve is *concave upwards* or *convex downwards* at P. [see fig (i)]

(ii) A portion of the curve on both sides of P, however small it may be, lies below the tangent PT (i.e. towards the negative direction of y-axis.)
In this case, we say that the curve is *concave downwards* or *convex upwards* at P. [see fig (ii).

(iii) It may be that the two portions of the curve on the two sides of P lie on different sides of the tangent PT, i.e. the curve crosses the tangent at P. In this case, we say that P is a *point of inflexion* on the curve [see fig. (iii)]

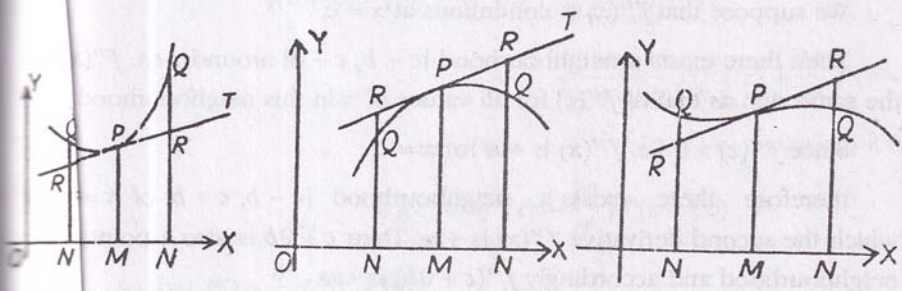
Note : The property of concavity or convexity depends on the choice of positive or negative side of y-axis which is fixed by convention. But this is not the case for a point of inflexion, which by definition, is a point where a curve crosses its tangent; its existence therefore, in no way depends on the choice of the axes.

8.2. Criteria for concavity, convexity and inflexion.

To determine whether a curve $y = f(x)$ is concave upwards, concave downwards or has a point of inflexion at $x = c$.

Let $P(x = c)$ be a point on the curve $y = f(x)$ and let Q be a point very near to P on the curve so that the x-co-ordinate of Q is $c + h$.

The point Q lies to the right or left of the point P according as h is positive or negative.



Draw QN perpendicular to the x axis meeting the tangent PT at P in R.

Now, the equation of the tangent at $P\{c, f(c)\}$ is

$$y - f(c) = f'(c)(x - c) \qquad \ldots(1)$$

The ordinate of R will be obtained by putting $x = c + h$ in (1), since the x-co-ordinate of R is also $= c + h$.

Hence, from (1) the ordinate of R i.e.

$$RN = f(c) + f'(c)(c + h - c)$$

i.e. $$RN = f(c) + hf'(c).$$

Also, the ordinate NQ of the curve for the abscissa $x = c + h$ is $f(c + h)$.

$\therefore$ $$RQ = NQ - NR = f(c + h) - f(c) - hf'(c).$$

(i) For concavity upwards at P (fig (i)) $NQ > NR$ i.e. RQ is $+ve$ when Q lies on either side of P.

(ii) For concavity downwards at P (fig (ii)) $NQ < NR$ i.e. RQ is $-ve$ when Q lies on either side of P.

(iii) For inflexion at P (fig (iii)), RQ is $+ve$ when Q lies on one side of P and $-ve$ when Q lies on other side of P.

Thus we have to examine the behaviour of RQ which from (1) is $f(c + h) - f(c) - hf'(c)$ for values of h which are sufficiently small numerically.

Now, by Taylor's Theorem with remainder after two terms, we have

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta h), \qquad \text{where } 0 < \theta < 1$$

$$\therefore \qquad RQ = \frac{h^2}{2!} f''(c + \theta h) \qquad \ldots(2)$$

Now we discuss the following three cases.

Case 1. Let $f''(c) > 0$.

We suppose that $f''(x)$ is continuous at $x = c$.

Then there exists a neighbourhood $[c - h, c + h]$ around c s.t. $f''(x)$ has the same sign as that of $f''(c)$ for all values of x in this neighbourhood.

Since $f''(c) > 0$ i.e. $f''(x)$ is $+ve$ for $x = c$,

therefore there exists a neighbourhood $[c - h, c + h]$ of $x = c$ for which the second derivative $f''(x)$ is $+ve$. Then $c + \theta h$ is also a point of this neighbourhood and accordingly $f''(c + \theta h)$ is $+ve$.

Also, $\frac{h^2}{2!}$ is $+ve$.

$\therefore$ $RQ > 0$ in the neighbourhood $[c - h, c + h]$.

Hence **the curve is concave upwards at P if $f''(c) > 0$.**

Case 2. Let $f''(c) < 0$.

Proceeding as above we find that $RQ < 0$ in the neighbourhood $[c - h, c + h]$.

Hence **the curve is concave downwards at P if $f''(c) < 0$.**

Case 3. Let $f''(c) = 0$ but $f'''(c) \neq 0$.

By Taylor's Theorem with remainder after three terms, we have

$$f(c + h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c) + \frac{h^3}{3!} f'''(c + \theta_1 h),$$

where $0 < \theta_1 < 1$

so that $RQ = \frac{h^3}{3!} f'''(c + \theta_1 h)$, since $f''(c) = 0$.

As before, there exists a neighbourhood $[c - h, c + h]$ around c such that for every point x in this neighbourhood $f'''(x)$ has the same sign as that of $f'''(c)$. Since $c + \theta_1 h$ lies in this neighbourhood, therefore $f'''(c + \theta_1 h)$ has the same sign as that of $f'''(c)$.

Also, $\frac{h^3}{3!}$ changes sign with the change in the sign of h.

Thus RQ changes sign with the changes in the sign of h.

Hence **the curve has inflexion at P if $f''(c) = 0$ and $f'''(c) \neq 0$.**

8.3. Generalisation.

Let $f''(c) = f'''(c) = ... = f^{n-1}(c) = 0$ but $f^n(c) \neq 0$.

By Taylor's Theorem with remainder after n terms, we have

$$f(c + h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c) + ...$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(c) + \frac{h^n}{n!} f^n(c + \theta h),$$

where $0 < \theta < 1$.

$$= f(c) + hf'(c) + \frac{h^n}{n!} f^n(c + \theta h).$$

so that $RQ = \frac{h^n}{n!} f^n(c + \theta h)$.

As we know, there exists a neighbourhood $[c - h, c + h]$ of c such that $f^n(c + \theta h)$ has the same sign as that of $f^n(c)$ for every point in this neighbourhood.

Also $\frac{h^n}{n!}$ changes its sign or keeps the same sign while the sign of h changes, according as n is odd or n is even.

Thus RQ changes sign if n is odd and keeps the sign of $f^n(c)$ if n is even.

Hence **if n is odd, the curve has inflexion at P;**

if n is even, it is concave upwards or downwards according as $f^n(c) > 0$ or $f^n(c) < 0$.

Note : We have already remarked that the position of the point of inflexion on a curve is independent of the choice of axes. so that in particular, the positions of x-axis and y-axis may be interchanged without affecting the positions of the points of inflexion on the curve. Thus the points of inflexion may also be determined by examining $\frac{d^2x}{dy^2}$ just as we examine $\frac{d^2y}{dx^2}$. For points where the tangent is parallel to y-axis i.e. where $\frac{dy}{dx}$ is infinite, it becomes necessary to examine $\frac{d^2x}{dy^2}$ instead of $\frac{d^2y}{dx^2}$.

EXAMPLE 1. *Show that $y = x^4$ is concave upwards at the origin.*

SOLUTION : From the given equation, we have

$$\frac{dy}{dx} = 4x^3, \quad \frac{d^2y}{dx^2} = 12x^2, \quad \frac{d^3y}{dx^3} = 24x, \quad \frac{d^4y}{dx^4} = 24.$$

At the origin (0, 0), $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$ but $\frac{d^4y}{dx^4}$ is $\neq 0$ and $+ve$.

Therefore the curve is concave upwards at the origin.

EXAMPLE 2. *For the curve $y = x^3$, show that $x = 0$ is a point of inflexion.*

SOLUTION : Here $\frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$, $\frac{d^3y}{dx^3} = 6$.

At the origin (0, 0), $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} \neq 0$.

Hence we have a point of inflexion at $x = 0$.

EXAMPLE 3. *Find if the curve $y = \log x$ is concave or convex upwards throughout.*

SOLUTION : Here $\frac{dy}{dx} = \frac{1}{x}$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

which is always negative.

Hence the curve is concave downwards i.e. convex upwards throughout.

EXAMPLE 4. *Find the range of values of x for which*

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$

is concave upwards or downwards. Also, determine its point of inflexion.

SOLUTION : Here $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$,

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24 = 12(x - 1)(x - 2).$$

Now, $\frac{d^2y}{dx^2} > 0 \Rightarrow 12(x - 1)(x - 2) > 0 \Rightarrow x < 1$ or $x > 2$

$$\frac{d^2y}{dx^2} < 0 \Rightarrow (x - 1)(x - 2) < 0 \Rightarrow 1 < x < 2.$$

Thus the curve is concave upwards in the intervals $(-\infty, 1)$, $(2, \infty)$ and is concave downwards in the interval (1, 2).

Also, $\frac{d^2y}{dx^2} = 0$ when $x = 1$ or 2, ...

But $\frac{d^3y}{dx^3} = 24x - 36 \neq 0$ for $x = 1$ or $x = 2$.

Therefore the given curve has inflexions at $x = 1$ and $x = 2$

From the given equation, when $x = 1$, we get $y = 19$ and when $x = 2$, $y = 33$. Hence (1, 19) and (2, 33) are the two points of inflexion on the curve.

EXAMPLE 5. *Find the points of inflexion, if any, of the curve* $y(a^2 + x^2) = x^3$.

SOLUTION : Here $y = \dfrac{x^3}{a^2 + x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{(a^2 + x^2).3x^2 - x^3(2x)}{(a^2 + x^2)^2}$$

$$= \frac{x^2(3a^2 + x^2)}{(a^2 + x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(a^2 + x^2)^2(3a^2.2x + 4x^3) - x^2(3a^2 + x^2).2(a^2 + x^2).2x}{(a^2 + x^2)^4}$$

$$= \frac{2a^2x(3a^2 - x^2)}{(a^2 + x^2)^3}; \text{ after simplification}$$

$$\frac{d^3y}{dx^3} = \frac{6a^2\{(x^2 - 3a^2)^2 - 9a^4 + a^2\}}{(a^2 + x^2)^4}.$$

Hence $\dfrac{d^2y}{dx^2} = 0$ when $x = 0, \pm \sqrt{3}\,a$ and we verify that $\dfrac{d^3y}{dx^3} \neq 0$ for these values of x. Hence $x = 0, \pm \sqrt{3}\,a$ are the three points of inflexion.

EXAMPLE 6. *Find the points of inflexion, if any of the curve* $x = (\log y)^3$.

SOLUTION : Here $\dfrac{dx}{dy} = \dfrac{3(\log y)^2}{y}$

$$\frac{d^2x}{dy^2} = 6\log y \cdot \frac{1}{y^2} - \frac{3}{y^2}(\log y)^2 = \frac{3\log y}{y^2}(2 - \log y)$$

$$\frac{d^3x}{dy^3} = \frac{6}{y^3} - \frac{12\log y}{y^3} - \frac{6\log y}{y^3} + \frac{6}{y^3}(\log y)^2$$

$$= \frac{6}{y^3} - \frac{18\log y}{y^3} + \frac{6}{y^3}(\log y)^2.$$

$\therefore \quad \dfrac{d^2x}{dy^2} = 0$ at $y = 1$ and $y = e^2$.

But $\left(\dfrac{d^3x}{dy^3}\right)_{y=1} = 6 \neq 0$ and $\left(\dfrac{d^3x}{dy^3}\right)_{y=e^2} = -\dfrac{6}{e^6} \neq 0.$

Hence $y = 1$ and $y = e^2$ give points of inflexion, so that (0, 1) and ($8, e^2$) are the two points of inflexion.

Example 34

1. Show that $y = e^x$ is everywhere concave upwards.
2. Show that the curve $y = x \log x$ is everywhere concave upwards.
3. Examine the curve $y = \sin x$ for concavity and convexity in the interval $(0, 2\pi)$
4. Show that the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards in $-2 < x < 0$ and $2 < x < \infty$ but concave downwards in $-\infty < x < -2$ and $0 < x < 2$ and at $x = -2, 0, +2$, there are points of inflexion.
5. Verify that the point of inflexion of the curve $y = xe^{-x}$ is at $x = 2$.
6. Show that the curve $y = e^{-x^2}$ has inflexions at $x = \pm \frac{1}{\sqrt{2}}$.
7. Satisfy yourself that the following curves have inflexions at the points indicated.

   (i) $y = x^2(3 - x)$; $(1, 2)$ (ii) $y = \frac{b^3}{a^2 + x^2}$; $\left(\pm \frac{a}{\sqrt{3}}, \frac{3b^3}{4a^2}\right)$

   (iii) $y = \frac{a^2 x}{(x - a)^2}$; $\left(-2a, -\frac{2}{9}a\right)$

   (iv) $y = \frac{a^2 x}{a^2 + x^2}$; $(\sqrt{3}\,a, \sqrt{3}\,a/4)$, $(0, 0)$, $(-\sqrt{3}\,a, -\sqrt{3}\,a/4)$.

   (v) $x = a\tan\theta$, $y = a\sin\theta\cos\theta$; $\left(\pm\sqrt{3}\,a, \pm\frac{\sqrt{3}}{4}a\right)$
8. Show that the curve $ay^2 = x(x - a)(x - b)$ has two and only two points of inflexion.
9. Show that the abscissae of the points of inflexion on the curve $y^2 = f(x)$ satisfy the equation $[f'(x)]^2 = 2f(x)f''(x)$.
10. Show that the curve $(1 + x^2)y = 1 - x$ has three points of inflexion and that they lie on a st. line.
11. Show that the points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the line $3x + a = 4b$.

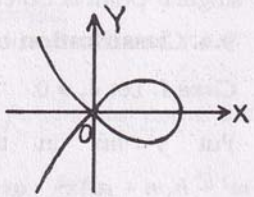
Answer

3. Concave downwards in $(0, \pi)$ and upwards in $(\pi, 2\pi)$.

9

SINGULAR POINTS

9.1. Introduction—The curves with implicit equations of the form $f(x, y) = 0$ exhibit some peculiarities which are not possessed by the curves with explicit equations of the form $y = g(x)$. These peculiarities arise from the fact that the equation $f(x, y) = 0$ may not define y as a single-valued function of x. In fact, to each value of x, there corresponds as many values of y as is the degree of the equation in y and these different values of y give rise to different branches of the curve. Thus we come across singularities on a curve occuring at points where two or more branches of the curve intersect. For example, we may consider the following graphs of the curve :



Here origin is a point common to the two branches of the curve.

9.2. Definitions

1. Double point—*A point through which there pass two branches of a curve is called a double point.*

A curve has two tangents at a double point, one for each branch of the curve.

The double point will be called a ***node,*** *a* ***cusp*** *or an* ***isolated point*** *according as the two tangents are different and real, coincident or imaginary.*

2. Triple point—*A point through which there pass 3 branches of a curve is called a triple point.*

3. Multiple point—*A point through which there pass r branches of a curve is called a multiple point of the rth order, so that a curve has r tangents at a multiple point of the rth order.*

A multipe point is also, sometimes called a **singular point.**

Cor. (i) A double point is a multiple point of the second order.

(ii) A triple point is a multiple point of the third order.

9.3. Singular points at the origin of an algebraic curve.

In order to investigate the nature of a singular point, it is necessary to find the tangents thereat. We recall here the results obtain in Art 6.7 in the chapter on "Curvature".

If a curve passes through the origin and is given by a rational integral equation, the equation to the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation of the curve.

When the curve passes through the origin, its equation may be put in the form :

$$(a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \ldots = 0$$

The origin is a singular point if at least $a_1 = b_1 = 0$.

If $a_1 = b_1 = 0$ but not of all a_2, b_2, c_2 are zero, the singular point is a **double point.**

If $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, but not all of a_3, b_3, c_3, d_3 are zero, then the singular point is called a **triple point** and so on.

9.4. Classification of double point at the origin.

Case I. Let $c_2 \neq 0$.

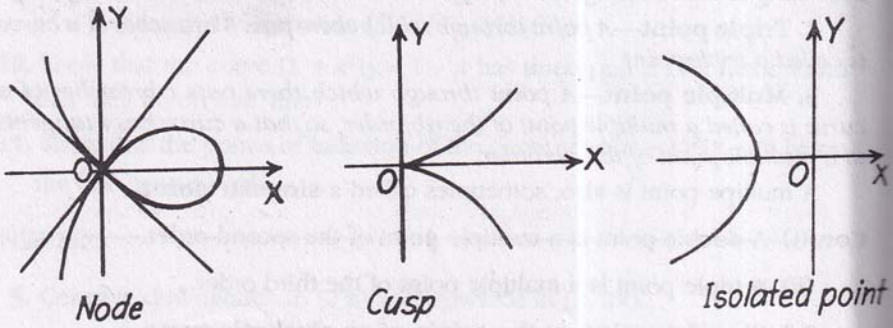
Put $y = mx$ in terms of $a_2x^2 + b_2xy + c_2y^2$ and we obtain $(c_2m^2 + b_2m + a_2)x^2$.

Now solve $c_2m^2 + b_2m + a_2 = 0$. Let the roots be m_1, m_2.

If the roots m_1, m_2 are real and distinct, the curve has two real and distinct tangents $y = m_1x$ and $y = m_2x$ at the origin and the double point is then called a *node* [fig. (i)].

If the roots are real and equal, the curve has generally two coincident tangents at the origin and the double point is then called a *cusp* [fig. (ii)].

If the roots are imaginary, i.e. if the two tangents are imaginary, the double point is called an *isolated point* [fig. (iii)].



Case 2. Let $c_2 = 0$, $a_2 \neq 0$.

Write $x = ny$ in terms $a_2x^2 + b_2xy$ of the curve and proceed as in case 1.

Case 3. Let $a_2 = 0 = c_2$, $b_2 \neq 0$.

The origin is then a node, the two tangents there being the co-ordinate axes.

EXAMPLE 1. *Examine the curve $x^3 + y^3 - 3axy = 0$ for singular points at the origin.*

SOLUTION : Clearly the curve passes through the origin and by equating to zero, the terms of the lowest degree we find that the equation of the tangent at the origin is $xy = 0$.

That is $x = 0$, $y = 0$ are the two tangents at the origin.

Hence the origin is a node on the given curve.

EXAMPLE 2. *Examine the curve $y^2(1 + x) = x^2(1 - x)$ for singular points at the origin.*

SOLUTION : The given equation can be written as

$$y^2 + xy^2 = x^2 - x^3 \quad \text{i.e.} \quad x^3 + xy^2 + y^2 - x^2 = 0.$$

Hence the tangents at the origin are

$$y^2 - x^2 = 0 \quad \text{i.e.} \quad y = x \quad \text{and} \quad y = -x.$$

Therefore origin is a node.

EXAMPLE 3. *Examine the curve $(x^2 + y^2)x - 2ay^2 = 0$ for singular points at the origin.*

SOLUTION : The lowest degree terms are of second degree; therefore origin is a double point. By equating to zero, the lowest degree terms, we find that the equation of the tangent at the origin is $y^2 = 0$ i.e. there is a single tangent $y = 0$.

Hence the origin is a cusp.

EXAMPLE 4. *Examine the curve $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$ for singular point at the origin.*

SOLUTION : The equations of the tangents at the origin are given by $a^2x^2 + b^2y^2 = 0$ i.e. $ax \pm iby$ are the two imaginary tangents at the origin.

Hence origin is an isolated point on the curve.

EXAMPLE 5. *Examine the curve $2y^5 + 5x^5 - 3x(x^2 - y^2) = 0$ for singular point at the origin*

SOLUTION : The lowest degree terms are of third degree; hence origin is a triple point.

The equations of the tangent at the origin are given by

$$x(x^2 - y^2) = 0 \text{ i.e. } x = 0,\ x = y,\ x = -y.$$

9.5. Singular points at points other than the origin.

For determining the existence and nature of the singular points at points other than the origin, we shall adopt two methods : one is by transferring the origin as we do in Co-ordinate Geometry and the other is by analytical method. However the second method is convenient to use. The first method is illustrated in this article and the second method in the next article.

If a point (h, k) be a singular point of the curve, (known in advance), we may shift the origin of the co-ordinate system to the point (h, k) by the well-known substitutions

$$x = x' + h;\ \ y = y' + k$$

where (x, y) are the current co-ordinates of the old system and (x', y') those of the new system. By so doing, the singular point is now $(0, 0)$ of the new co-ordinate system. Now, we may use the rules suggested in the previous article for determining the nature of the singular point.

EXAMPLE 1. *Determine the existence and nature of the double points on the curve* $(x - 2)^2 = y(y - 1)^2$.

SOLUTION : We transfer the origin to a point (h, k), so that as usual we put $x = x' + h$ and $y = y' + k$.

The given equation becomes

$$(x' + h - 2)^2 = (y' + k)(y' + k - 1)^2$$

$$\Rightarrow \quad x'^2 + 2(h - 2)x' + (h - 2)^2 = (y' + k)\{y'^2 + 2(k - 1)y' + (k - 1)^2\}$$

$$= y'^3 + 2(k - 1)y'^2 + (k - 1)^2 y' + ky'^2 + 2k(k - 1)y' + k(k - 1)^2$$

$$\Rightarrow \quad y'^3 + 2(k - 1)y'^2 + ky'^2 - x'^2 + (k - 1)^2 y' + 2k(k - 1)y' - 2(h - 2)x' + k(k - 1)^2 - (h - 2)^2 = 0$$

$$\Rightarrow \quad y'^3 + y'^2\{2(k - 1) + k\} - x'^2 + y'\{(k - 1)^2 + 2k(k - 1)\} - 2(h - 2)x' + k(k - 1)^2 - (h - 2)^2 = 0 \qquad ...(1)$$

Hence for a double point, we must have,

$$k(k - 1)^2 - (h - 2)^2 = 0 \qquad ...(2)$$

$$h - 2 = 0 \qquad ...(3)$$

$$(k - 1)^2 + 2k(k - 1) = 0 \qquad ...(4)$$

Solving (2), (3) and (4) we get $h = 2$, $k = 1$.

Therefore (2, 1) is a double point of the curve.

Transferring the origin to (2, 1), we get from (1) $x'^2 = y'^2(y' + 1)$.

Tangents at the origin are $x'^2 = y'^2$ i.e. $y' = \pm x'$.

Hence there is a node at (2, 1) where the two tangents, according to the old co-ordinate system, are $y - 1 = \pm (x - 2)$.

9.6. Conditions for the existence of multiple points.

Let $P(x, y)$ be a point on the curve $f(x, y) = 0$.

We have seen in chapter 4 on 'tangent and normal' that at a point (x, y) of the curve $f(x, y) = 0$, the slope of the tangent is given by $\frac{dy}{dx}$ where

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

i.e. $$f_x + f_y \frac{dy}{dx} = 0 \qquad \ldots(1)$$

At a multiple point of a curve, the curve has at least two tangents and accordingly $\frac{dy}{dx}$ must have at least two values at a multiple point.

The equation (1) being of the first degree in $\frac{dy}{dx}$ can be satisfied by more than one value of $\frac{dy}{dx}$, if and only if $f_x = 0$, $f_y = 0$. so that (1) becomes an identity.

Therefore, in order to find the multiple points, we must have such values of x and y which simultaneously satisfy the three equations

$$f(x, y) = 0, \; f_x(x, y) = 0, \; f_y(x, y) = 0.$$

9.7. To find the slopes of the tangents at a double point.

Differentiating (1) w.r.t. x, we get

$$\frac{d}{dx}\left\{\frac{\partial f}{\partial x}\right\} + \frac{d}{dx}\left\{\frac{\partial f}{\partial y} \cdot \frac{dy}{dx}\right\} = 0 \qquad \ldots(2)$$

First we evaluate $\frac{d}{dx}\left\{\frac{\partial f}{\partial x}\right\}$.

For this, let $\frac{\partial f}{\partial x} = u$.

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

so that $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left\{\frac{\partial f}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\frac{\partial f}{\partial x}\right\} + \frac{dy}{dx} \cdot \frac{\partial}{\partial y}\left\{\frac{\partial f}{\partial x}\right\}$$

i.e. $$\frac{d}{dx}(f_x) = \frac{\partial}{\partial x}(f_x) + \frac{dy}{dx} \cdot f_{yx}$$

i.e. $$= f_{xx} + \frac{dy}{dx} \cdot f_{yx}.$$

Similarly $$\frac{d}{dx}\left[f_y \frac{dy}{dx}\right] = f_y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx}\{f_y\}$$

$$= f_y \frac{d^2y}{dx^2} + \frac{dy}{dx}\left[f_{xy} + f_{yy}\frac{dy}{dx}\right].$$

Hence from (2),

$$f_{xx} + f_{xy}\frac{dy}{dx} + \left(f_{yx} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0,$$

so that at a double point where $f_x = 0, f_y = 0$, the values of $\frac{dy}{dx}$ are the roots of the quadratic equation.

$$f_{xx} + f_{xy}\frac{dy}{dx} + f_{yx}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + 0 = 0$$

i.e. $$f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\left(\frac{dy}{dx}\right) + f_{xx} = 0. \qquad \ldots(3)$$

Hence, in case if f_{xx}, f_{xy}, f_{yy} are not all zero and $f_x = 0 = f_y$, the point (x, y) will be a double point and will be a node, cusp or conjugate according as the values of $\frac{dy}{dx}$ are real and distinct, equal or imaginary i.e. according as the discriminant of (3) viz,

$$(f_{xy})^2 - f_{xx}f_{yy} > 0, = 0 \text{ or } < 0.$$

If $f_{xx} = f_{xy} = f_{yy} = 0$, the point (x, y) will be a multiple point of order higher than two.

EXAMPLE 1. *Determine the position and nature of the multiple points of the curve* $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$. *Also, find the tangents at the multiple points if any.*

SOLUTION : Here $f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13$

Then $f_x = 3x^2 - 14x + 15$; $f_y = -2y + 4$.

$\therefore \quad f_x = 0 \Rightarrow 3x^2 - 14x + 15 = 0$

$\Rightarrow (x-3)(3x-5) = 0 \Rightarrow x = 3 \text{ or } 5/3$

and $f_y = 0 \Rightarrow -2y + 4 = 0 \Rightarrow y = 2.$

Therefore $f_x = 0$ and $f_y = 0$ for the points (3, 2) and (5/3, 2).

Of these two points, only the point (3, 2) lies on the curve.

Hence this is the only multiple point.

Again, $f_{xy}(3, 2) = 0, \quad f_{yx}(3, 2) = 0$

$f_{xx} = 6x - 14 \quad \therefore \quad f_{xx}(3, 2) = 4$

$f_{yy} = -2 \quad \therefore \quad f_{yy}(3, 2) = -2.$

$\therefore (f_{xy})^2 - f_{xx} f_{yy} = 0 - 4(-2) = 8 > 0$ and f_{xy}, f_{xx}, f_{yy} are not all zero.

Hence (3, 2) is a double point and there is a node at (3, 2).

The slopes of the tangents are given by

$$f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\left(\frac{dy}{dx}\right) + f_{xx} = 0$$

$$\Rightarrow \quad -2\left(\frac{dy}{dx}\right)^2 + 2 \times (0) + 4 = 0 \Rightarrow \left(\frac{dy}{dx}\right)^2 = 2 \quad \therefore \quad \frac{dy}{dx} = \pm\sqrt{2}.$$

Hence the equations of tangents at the node (3, 2) are given by

$$y - 2 = \pm\sqrt{2}(x - 3).$$

Second Method : To find the tangents at (3, 2) we shift the origin to this point. For this, we put $x = x' + 3, \ y = y' + 2$ where (x, y) are the current co-ordinates of the old system and (x', y') those of the new system.

With this change, the equation becomes

$$(x'+3)^3 - (y'+2)^2 - 7(x'+3)^2 + 4(y'+2) + 15(x'+3) - 13 = 0$$

$$\Rightarrow \quad \{x'^3 + 3x'^2(3) + 3x'(9) + 27\} - (y'^2 + 4y' + 4) - 7(x'^2 + 6x' + 9)$$
$$+ 4(y' + 2) + 15(x' + 3) - 13 = 0$$

$$\Rightarrow \quad x'^3 + 2x'^2 - y'^2 = 0.$$

$\therefore$ The tangents at the new origin are

$$2x'^2 - y'^2 = 0 \Rightarrow y' = \pm\sqrt{2}x'.$$

This is obtained by putting to zero, the terms of the lowest degree.

Reverting back to the old system, the equations of the tangent at (3, 2) are

$$y - 2 = \pm\sqrt{2}(x - 3), \text{ as before}$$

Note : The first method is quite convenient.

EXAMPLE 2. *Determine the position and nature of the double points on the curve* $y(y-6) = x^2(x-2)^3 - 9$. **(Gorakhpur 88, 90)**

SOLUTION : Let $f(x, y) = y(y-6) - x^2(x-2)^3 + 9$

$$= y^2 - 6y - x^2(x^3 - 6x^2 + 12x - 8) + 9$$

$$= y^2 - 6y - x^5 + 6x^4 - 12x^3 + 8x^2 + 9$$

Then $f_x = -5x^4 + 24x^3 - 36x^2 + 16x$; $f_y = 2y - 6$.

Now $f_x = 0 \Rightarrow 5x^4 - 24x^3 + 36x^2 - 16x = 0$

$\Rightarrow x(5x^3 - 24x^2 + 36x - 16) = 0$

$\Rightarrow x(x-2)(5x^2 - 14x + 8) = 0$

$\Rightarrow x(x-2)(x-2)(5x-4) = 0$

$\Rightarrow x = 0,\ 2,\ 4/5.$

$f_y = 0 \Rightarrow 2y - 6 = 0 \Rightarrow y = 3.$

Therefore $f_x = 0$ and $f_y = 0$ for the points (0, 3), (2, 3) and (4/5, 3).

Of these three points, only the points (0, 3), (2, 3) lie on the curve.

Hence the multiple points are (0, 3) and (2, 3).

Again $f_{xx} = -20x^3 + 72x^2 - 72x + 16$

$f_{xy} = 0;\ f_{yy} = 2.$

At (0, 3) $f_{xx} = 16,\ f_{xy} = 0,\ f_{yy} = 2.$

At (2, 3), $f_{xx} = 0,\ f_{xy} = 0, f_{yy} = 2.$

At (0, 3), $(f_{xy})^2 - f_{xx}f_{yy} = 0 - 16 \times 2 = -32 < 0.$

$\therefore$ (0, 3) is a conjugate point.

At (2, 3), $(f_{xy})^2 - f_{xx}f_{yy} = 0 - 0 \times 2 = 0$

$\therefore$ (2, 3) is a cusp.

The slopes of the tangents are given by

$$f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\left(\frac{dy}{dx}\right) + f_{xx} = 0$$

$$\Rightarrow \quad 2\left(\frac{dy}{dx}\right)^2 + 2(0) + 0 = 0 \Rightarrow \frac{dy}{dx} = 0.$$

Hence the equation of the tangent at the cusp (2, 3) is given by

$$y - 3 = 0(x - 2) \text{ i.e. } y = 3.$$

Example 35

Find the position and nature of the multiple points on the following curves.

1. $x^3 + y^3 - 3axy = 0$ **2.** $x^2(x-y) + y^2 = 0$

3. $x^3 + y^3 = 6x^2$ **4.** $y^3 = x^3 + ax^2$

5. $y^2(x-1) = x^3$ **6.** $y^2(x^2-4) = x^4$

7. $y^2 = (x-1)(x-2)^2$ **8.** $x^4 + y^3 - 2x^2 + 3y^2 = 0$

9. $xy^2 - ax^2 + 2a^2x - a^3 = 0$

10. $x^4 - ax^2y + axy^2 + a^2y^2 = 0.$

11. $x^4 - 4ax^3 + 2ay^3 + 4a^2x^2 - 3a^2y^2 - a^4 = 0$ **(Gorakhpur. 89)**

12. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$

13. Show that the origin is a node, cusp or an isolated point on the curve $y^2 = ax^2 + bx^3$ according as $a > 0$, $=$ or < 0.

14. Show that $(a,0)$ is a node, cusp or an isolated point on the curve $ay^2 = (x-a)^2(x-b)$ according as $b/a <$, $=$ or >1 .

15. Determine the position and nature of the multiple points on the curve $x^2y^2 = (a+y)^2(b^2-y^2)$ distinguishing between the cases $b <$, $=$, $> a$

Find the equations of the tangents at the multiple points of the following curves :

16. $y^2(1+x) = x^2(1-x)$ **17.** $(x-2)^2 = y(y-1)^2$

18. $x^4 + y^3 - 2x^2 + 3y^2 = 0$ **19.** $(y-2)^2 = x(x-1)^2$

20. $(x+y)^3 - \sqrt{2}(y-x+2)^2 = 0$

ANSWERS

1. (0, 0) double point , node **2.** cusp at (0, 0)

3. (0, 0) double point , cusp **4.** cusp at (0, 0)

5. double point, cusp **6.** isolated double point

7. node at (2, 0) **8.** node at (0, 0)

9. node at $(a, 0)$ **10.** isolated double point

11. isolated point at $(a, 0)$ **12.** nodes at $(a, 0)$, $(-a, 0)$, $(0, -a)$

15. $(0, -a)$ is a double point , a node, cusp or isolated point according as $b > a$, $b = a$ or $b < a$.

16. $y = \pm x$, origin node **17.** $y - 1 = \pm (x-2)$, node at (2, 1)

18. node at (0, 0), $y = \pm \sqrt{\frac{2}{3}}x$ **19.** $x - y + 1 = 0$, $x + y = 3$

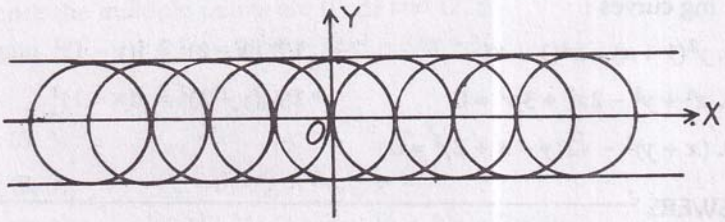
20. cusp at (1, – 1); $y = x - 2$.

10

ENVELOPES

10.1. Families of curves—The equation of a curve often involves, besides the current co-ordinates x and y, certain constants upon which the shape, size and position of a particular curve depend in relation to the reference system.

For example, $(x - \alpha)^2 + y^2 = a^2$ represents a circle whose centre lies on the x-axis at a distance of α from the origin. For fixed a, its position depends on the radius α. Suppose we fix a and allow α to take on a series of values, then we have a series of circles of equal radii a differing in positions on the centre. We shall have a picture like this:



Similarly if $f(x, y, \alpha)$ be any function of three variables, then the equation $f(x, y, \alpha) = 0$ determines a curve corresponding to each particular value of α. The totality of these curves, obtained by assigning different values to α is said to be a *family of curves.*

The variable α which is constant for one particular curve but changes in passing from one curve to another, i.e. the variable α, which is different for different curves, is said to be the *parameter* for the family.

Note : Since $f(x, y, \alpha) = 0$ involves only one parameter, therefore it is called one parameter family of curves.

For example,

(i) The equation $x^2 + y^2 - 2ax = 0$ represents a family of curves which are circles with their centres $(a, 0)$ on the x-axis and which pass through the origin. Here a is the parameter.

(ii) The equation $y = mx + \frac{a}{m}$ determines a family of st. lines which are tangents to the parabola $y^2 = 4ax$.
Here the parameter is m.

10.2. Envelope :

Definition—*The envelope of one—parameter family of curves is the locus of the limiting position of the points of intersection of the neighbouring curves as the parameter varies continuously.*

10.3. Determination of envelope

Let $f(x, y, \alpha) = 0$ ...(1) be any given family of curves.

Consider any two members

$$f(x, y, \alpha) = 0$$

and $$f(x, y, \alpha + \delta\alpha) = 0 \qquad ...(2)$$

corresponding to the parametric values α and $\delta\alpha$.

The points common to the two curves (1) and (2) satisfy the equation

$$f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha) = 0$$

$$\Rightarrow \quad \frac{f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha)}{\delta\alpha} = 0 \qquad ...(3)$$

Now, let $\delta\alpha \to 0$.

Then $$\underset{\delta\alpha \to 0}{Lt} \frac{f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha)}{\delta\alpha} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial\alpha} f(x, y, \alpha) = 0$$

That is, the limiting position of the point of intersection of (1) and (2) satisfy the equation

$$f_\alpha(x, y, \alpha) = 0 \qquad ...(4)$$

Thus the co-ordinates of the points on the envelope satisfy the equations (1) and (4).

Let the elimination of α between (1) and (4) lead to an equation $\phi(x, y) = 0$. That is, the required envelope is $\phi(x, y) = 0$.

Rule : To obtain the envelope of family of curves, its equation may be obtained in either of the following ways:

1. Eliminate α between

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial}{\partial\alpha} f(x, y, \alpha) = 0 \qquad ...(A)$$

The α - eliminant is the reqd. envelope.

2. Solve for x and y in terms of α from the equation (A).
It will give the parametric representation of the envelope, α being the parameter.

Note : The equation obtained by eliminating α between $f = 0$ and $f_\alpha = 0$ is exactly the condition that the relation $f(x, y, \alpha) = 0$ considered as an equation in α has a repeated root. Thus if

$$f(x, y, \alpha) \equiv A(x, y)\alpha^2 + B(x, y)\alpha + C(x, y) = 0.$$

then its envelope is obtained by the use of the condition that the above quadratic equation in α, will have its roots equal. Thus its envelope is given by $B^2 - 4AC = 0$. **(Gorakhpur 92)**

EXAMPLE : *Obtain the envelope of the family of lines* $y = mx + \frac{a}{m}$ *(a fixed, m a variable parameter)*

SOLUTION : The given equation can be written as:

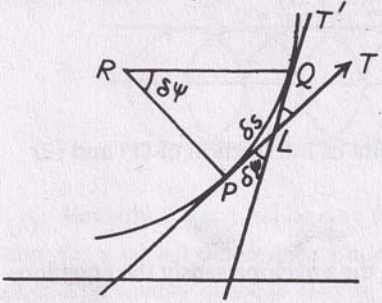
$$my = m^2x + a \Rightarrow m^2x - my + a = 0$$

which is a quadratic in m.

The condition that this equation has its two roots equal is $y^2 - 4ax = 0$ (parabola) which gives the required envelope.

10.4. *Theorem*

The evolute of a curve is the envelope of its normals.



Proof : Let PT, QT' be the tangents and PR and QR be the normals at two points P and Q of a curve.

Let L be the point of intersection of the tangents.

Then as usual $\angle PRQ = \delta\psi$ and arc $PQ = \delta s$.

From property of ΔPRQ,

$$\frac{PR}{\sin(RQP)} = \frac{PQ}{\sin(PRQ)}$$

$$\Rightarrow \quad PR = \frac{PQ}{\sin\delta\psi}\sin(RQP)$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\sin\,\delta\psi}\sin(RQP)$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{d\psi}{\sin\delta\psi}\sin(RQP)$$

Let $Q \to P$ so that $\angle RQP \to \angle RPT = \pi/2$.

$$\therefore \quad \underset{Q \to P}{Lt} PR = 1 \cdot \rho \cdot 1 \cdot \sin\frac{\pi}{2} = \rho.$$

Thus the limiting position of R which is the intersection of the normals at P and Q is the centre of curvature at P.

Hence the theorem.

10.5. Geometrical relation between a family of curves and its envelope.

Theorem I. *To prove that any singular point of any curve of a given family is a point on its envelope.*

Proof: Let $f(x, y, \alpha)$ be a family of curves. We have to show that if for any point on any member of the family

$$f(x, y, \alpha) = 0 \qquad ...(1)$$

we have $f_x = f_y = 0$,

then for such a point, we will also have $f_\alpha = 0$.

From (1), we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial \alpha}d\alpha = 0$$

Since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, therefore we get $f_\alpha = 0$.

Thus the locus of the singular points of the curves of a family is a part of its envelope.

Hence the result.

Theorem II. *To prove that, in general, the envelope of a family of curves touches each member of the family.*

Proof. Let $f(x, y, \alpha) = 0$ ...(1)

be a given family of curves. Its envelope is obtained by eliminating α between (1) and $f_\alpha(x, y, \alpha) = 0$ ...(2)

Let $x = \phi(\alpha), \quad y = \psi(\alpha)$ ...(3)

be the parameteric equation of the envelope obtained by solving (1) and (2) for x and y in terms of α.

The equations (3) satisfy the equation (1) identically i.e. for every value of α.

We differentiate (1) w.r.t. α regarding x, y as functions of α and we obtain

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{d\alpha} + \frac{\partial f}{\partial y}\cdot\frac{dy}{d\alpha} + \frac{\partial f}{\partial \alpha} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\phi'(\alpha) + \frac{\partial f}{\partial y}\psi'(\alpha) + 0 = 0; \text{ because of (2)}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\phi'(\alpha) + \frac{\partial f}{\partial y}\psi'(\alpha) = 0. \qquad \ldots(4)$$

The slope of tangent at an ordinary point (x, y) of a curve 'α' of the family is $-\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$.

Also, the slope of the tangent at the same point to the envelope (3) is $\psi'(\alpha)/\phi'(\alpha)$.

We see from (4) that these two slopes are the same. Thus the slopes of the tangents to the curve and envelope at the common point are equal. This means that the curve and the envelope have the same tangent at the common point and hence they touch each other.

Note : If for any point on the curve $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero, then the above argument will break down, so that the envelope may not touch a curve at points which are singular points on the curve.

Cor. We know that a st. line and a conic cannot have a singular point. Hence the envelope of a family of st. lines or of conics touches each member of the family at all their common points without exception.

Worked out Examples

EXAMPLE 1. *Find the envelope of the family of st. lines* $y = mx + am^3$, *m being the parameter.* **(Gorakhpur 92)**

SOLUTION : Given : $y = mx + am^3$ ...(1)

Differentiating (1) w.r.t. m, we get

$$0 = x + a.3\ m^2 \Rightarrow m^2 = -\frac{x}{3a}. \qquad \ldots(2)$$

Now, we have to eliminate m between (1) and (2).

From (1), $y = mx + am^2.m$

$$= mx + am \cdot \left(-\frac{x}{3a}\right) = m\left(x - \frac{x}{3}\right)$$

$$= \frac{2x}{3}m$$

$$\Rightarrow \quad y^2 = \frac{4x^2}{9}m^2 = \frac{4x^2}{9}\left(-\frac{x}{3a}\right) = -\frac{4x^3}{27a}$$

$\therefore \quad 27ay^2 + 4x^3 = 0$

which is the required envelope.

EXAMPLE 2. *Find the envelope of the family of semi-cubical parabolas*

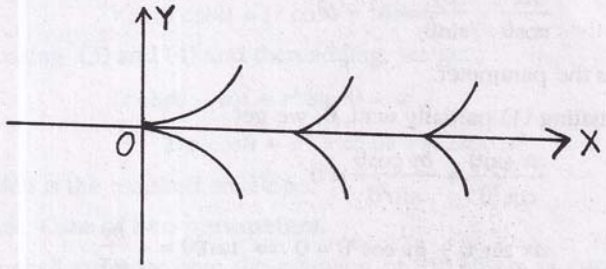
$$y^2 - (x + \alpha)^3 = 0. \qquad ...(1)$$

SOLUTION : Differentiating (1) w.r.t. α, we get

$$3(x + \alpha)^2 = 0 \qquad ...(2)$$

Eliminating α between (1) and (2), we get $y = 0$ which is the required envelope.

Note : It can be verified that $(-\alpha, 0)$ is a singular point. Thus $y = 0$ is the locus of singular points i.e. cusps of (1) and also it touches each member of the family.



Family of semi-cubical parabolas.

EXAMPLE 3. *Obtain the envelope of the circles drawn upon the radii vectors of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ as diameter.}$$

SOLUTION : Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse.

Then the equation of the circle on the radius vector OP as the diameter is

$$(x - 0)(x - a\cos\theta) + (y - 0)(y - b\sin\theta) = 0$$

i.e. $$x^2 + y^2 - ax.\cos\theta - by\sin\theta = 0. \qquad ...(1)$$

Differentiating w.r.t. θ, we get

$$ax\sin\theta - by\cos\theta = 0 \Rightarrow \tan\theta = \frac{by}{ax}.$$

Whence $$\sin\theta = \frac{by}{\sqrt{a^2x^2 + b^2y^2}} \text{ and } \cos\theta = \frac{ax}{\sqrt{a^2x^2 + b^2y^2}}.$$

Putting the values of $\sin\theta$ and $\cos\theta$ in (1), we get

$$x^2 + y^2 - ax \cdot \frac{ax}{\sqrt{a^2x^2 + b^2y^2}} - by \cdot \frac{by}{\sqrt{a^2x^2 + b^2y^2}} = 0$$

$$\Rightarrow \qquad x^2 + y^2 - \frac{a^2x^2 + b^2y^2}{\sqrt{a^2x^2 + b^2y^2}} = 0$$

$$\Rightarrow \qquad x^2 + y^2 = \sqrt{a^2x^2 + b^2y^2} \Rightarrow (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

which is the required envelope.

EXAMPLE 4. *Considering the evolute of a curve as the envelope of its normals, find the evolute of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION : The equation of the normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad ...(1)$$

Here θ is the parameter.

Differentiating (1) partially w.r.t. θ, we get

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0$$

$$\Rightarrow \qquad ax\sin^3\theta + by\cos^3\theta = 0 \Rightarrow \tan^3\theta = -\frac{by}{ax}$$

$$\Rightarrow \tan\theta = -\left(\frac{by}{ax}\right)^{1/3} \qquad ...(2)$$

To obtain the envelope, we need to eliminate θ between (1) and (2).

From (2), $\sin\theta = \mp \frac{(by)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}}$, $\cos\theta = \pm \frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}}$.

Substituting these values in (1), we get

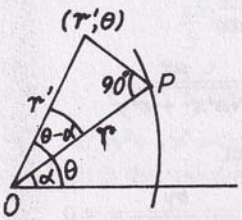
$$\pm [(ax)^{2/3} + (by)^{2/3}]\,[(ax)^{2/3} + (by)^{2/3}]^{1/2} = a^2 - b^2$$

$$\Rightarrow \qquad \pm [(ax)^{2/3} + (by)^{2/3}]^{3/2} = a^2 - b^2$$

$$\Rightarrow \qquad (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

which is the required envelope.

EXAMPLE 5. *Find the envelope of st. lines drawn at right angles to the radius vectors of the cardiode* $r = a(1 + \cos\theta)$, *through their extremities.*



SOLUTION : Let P be any point on the curve and let α be its vectorial angle. Then its radius vector $OP = a(1 + \cos\alpha)$.

The equation of the line drawn through P at right angles to the radius vector OP is

$$r\cos(\theta - \alpha) = a(1 + \cos\alpha) \qquad ...(1$$

The angle α is different for different radii vectors.

Differentiating (1) w.r.t. α, we get

$$-r\sin(\theta-\alpha)(-1) = -a\sin\alpha$$

i. e. $$r\sin(\theta-\alpha) = -a\sin\alpha \quad ...(2)$$

Now, we have to eliminate α from (1) and (2).

From (1), $$r(\cos\theta.\cos\alpha + \sin\theta\sin\alpha) = a + a\cos\alpha$$

$\Rightarrow$ $$(r\cos\theta - a)\cos\alpha + r\sin\theta\sin\alpha = a \quad ...(3)$$

From (2), $$r(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = -a\sin\alpha$$

$\Rightarrow$ $$r\sin\theta\cos\alpha = (r\cos\theta - a)\sin\alpha \quad ...(4)$$

Squaring (3) and (4) and then adding, we get

$$(r\cos\theta - a)^2 + r^2\sin^2\theta = a^2$$

$\Rightarrow$ $$r^2 - 2ra\cos\theta + a^2 = a^2 \Rightarrow r = 2a\cos\theta$$

which is the required envelope.

10.6. Case of two parameters.

We shall consider here the equation of the family of curves which involve two parameters α, β connected by a given relation.

Let $$f(x, y, \alpha, \beta) = 0 \quad ...(1)$$

be the equation of a curve and let

$$\phi(\alpha, \beta) = 0 \quad ...(2)$$

be the relation connecting the parameters α and β.

In order to determine its envelope, we may eliminate one of the parameters by means of the equation (2) and thus the problem is reduced to that solved before. However, the following method is more convenient..

For a fixed point (x, y) of the envelope, we have, from (1) and (2), by differentiation,

$$\frac{\partial f}{\partial\alpha} + \frac{\partial f}{\partial\beta}\cdot\frac{d\beta}{d\alpha} = 0 \quad ...(3)$$

$$\frac{\partial\phi}{\partial\alpha} + \frac{\partial f}{\partial\beta}\cdot\frac{d\beta}{d\alpha} = 0 \quad ...(4)$$

Eliminating $\frac{d\beta}{d\alpha}$ from (3) and (4), we get.

$$\frac{\partial f}{\partial\alpha} \Big/ \frac{\partial\phi}{\partial\alpha} = \frac{\partial f}{\partial\beta} \Big/ \frac{\partial\phi}{\partial\beta} \quad ...(5)$$

We now , eliminate α, β from (1), (2) and (5). The eliminant gives the required envelope.

Worked out Examples

EXAMPLE. 1. *Find the envelope of the family of lines*

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(1)$$

where the parameters a and b are connected by the relation.

(i) $a^n + b^n = c^n$ **(Gorakhpur 89)** ...(2)

(ii) $ab = c^2$ **(Gorakhpur 91)** ...(3)

SOLUTION : (i) We consider b as a function of a determined from (2).

Differentiating (1) and (2) w.r.t. a, we get

$$x\left(-\frac{1}{a^2}\right) + y\left(-\frac{1}{b^2}\right)\frac{db}{da} = 0;$$

$$na^{n-1} + nb^{n-1}\frac{db}{da} = 0.$$

Eliminating $\frac{db}{da}$ from both the equations, we get

$$\frac{-x/a^2}{na^{n-1}} = \frac{y/b^2}{-nb^{n-1}}$$

$$\Rightarrow \quad \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} \quad \Rightarrow \quad \frac{x/a}{a^n} = \frac{y/b}{b^n} = \frac{x/a + y/b}{a^n + b^n} = \frac{1}{c^n}; \quad \text{from (1) and (2)}$$

$$\Rightarrow \quad a^{n+1} = c^n x \quad \text{and} \quad b^{n+1} = c^n y$$

$$\Rightarrow \quad a = c^{\frac{n}{n+1}} \cdot x^{\frac{1}{n+1}} \quad \text{and} \quad b = c^{\frac{n}{n+1}} y^{\frac{1}{n+1}}$$

$$\Rightarrow \quad a^n = c^{\frac{n^2}{n+1}} x^{\frac{n}{n+1}} \quad \text{and} \quad b^n = c^{\frac{n^2}{n+1}} y^{\frac{n}{n+1}}$$

Adding, we get.

$$a^n + b^n = c^{\frac{n^2}{n+1}}\left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}\right)$$

$$\Rightarrow \quad c^n = \text{" " " " "}$$

$$\therefore \quad x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{n - \frac{n^2}{n+1}} = c^{\frac{n}{n+1}}.$$

which is the required envelope.

Cor. (A) Putting $n = 1$, i. e. when $a + b = c$, we get the reqd. envelope as $\sqrt{x} + \sqrt{y} = \sqrt{c}$.

(B) Putting $n = 2$, i. e when $a^2 + b^2 = c^2$, the reqd. envelope is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

(ii) Differentiating (1) and (3) w.r.t. a, we get

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0;$$

and $$b + a\frac{db}{da} = 0.$$

Eliminating $\frac{db}{da}$ from both the equations, we get

$$\frac{x}{a^2 b} = \frac{y}{b^2 a} \Rightarrow \frac{x}{a} = \frac{y}{b}$$

$$\Rightarrow \quad \frac{x}{a} = \frac{y}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{2} = \frac{1}{2}$$

$$\Rightarrow \quad a = 2x \text{ and } b = 2y \Rightarrow ab = 4xy$$

$$\Rightarrow \quad 4xy = c^2.$$

which is the reqd. envelope

EXAMPLE 2. *Show that the envelope of the family of parabolas* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ *under the condition*

(i) $a + b = c$ *is* $x^{1/3} + y^{1/3} = c^{1/3}$ **(Gorakhpur 87)**

(ii) $ab = c^2$ *is a hyperbola having its asymptotes coinciding with axes.*

SOLUTION : (i) Given $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ ...(1)

and $$a + b = c$$ ...(2)

Differentiating (1) and (2) w.r.t. a, we get

$$\sqrt{x}\left(-\frac{1}{2a^{3/2}}\right) + \sqrt{y}\left(-\frac{1}{2b^{3/2}}\right)\frac{db}{da} = 0;$$

$$1 + \frac{db}{da} = 0.$$

Eliminating $\frac{db}{da}$ from both the equations, we get

$$\frac{\sqrt{x}}{2a^{3/2}} = \frac{\sqrt{y}}{2b^{3/2}}$$

$$\Rightarrow \quad \frac{\sqrt{x/a}}{a} = \frac{\sqrt{y/b}}{b} = \frac{\sqrt{x/a} + \sqrt{y/b}}{a+b} = \frac{1}{c}$$

$$\Rightarrow \quad \frac{\sqrt{x}}{a^{3/2}} = \frac{\sqrt{y}}{b^{3/2}} = \frac{1}{c}$$

$$\Rightarrow \quad a^{3/2} = c\sqrt{x} \text{ and } b^{3/2} = c\sqrt{y}$$

$$\Rightarrow \quad a = c^{2/3}x^{1/3} \text{ and } b = c^{2/3}y^{1/3}$$

On adding, we get

$$a + b = c^{2/3}(x^{1/3} + y^{1/3})$$

$$\Rightarrow \quad c = \text{,,} \quad \text{,,} \quad \text{,,}$$

$$\therefore \quad x^{1/3} + y^{1/3} = c^{1/3}.$$

Hence the result.

(ii) Given $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ ...(3)

and $ab = c^2$ ...(4)

Differentiating (3) and (4) w.r.t. a, we get

$$\sqrt{x}\left(-\frac{1}{2a^{3/2}}\right) + \sqrt{y}\left(-\frac{1}{2b^{3/2}}\right)\frac{db}{da} = 0$$

and $$b + a\frac{db}{da} = 0.$$

Eliminating $\frac{db}{da}$ from both the equations, we get

$$\frac{\sqrt{x}}{2a^{3/2}b} = \frac{\sqrt{y}}{2ab^{3/2}}$$

$$\Rightarrow \quad \frac{\sqrt{x}}{\sqrt{a}} = \frac{\sqrt{y}}{\sqrt{b}} = \frac{\sqrt{x/a} + \sqrt{y/b}}{2} = \frac{1}{2}$$

$$\Rightarrow \quad \sqrt{a} = 2\sqrt{x} \text{ and } \sqrt{b} = 2\sqrt{y}$$

$$\Rightarrow \quad \sqrt{a.b} = 4\sqrt{xy} \Rightarrow 16xy = c^2.$$

Hence the result.

EXAMPLE 3. *Find the envelope of the family of coaxial ellipses*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(1)$$

when the parameters a and b are connected by $a^n + b^n = c^n$ ...(2)

SOLUTION : We consider b as a function of a determined from (2).

Differentiating (1) and (2) w.r.t. a, we get

$$x^2\left(-\frac{2}{a^3}\right)+y^2\left(-\frac{2}{b^3}\right)\frac{db}{da}=0$$

and $$na^{n-1}+nb^{n-1}\frac{db}{da}=0.$$

Eliminating $\frac{db}{da}$, we get

$$\frac{-2x^2}{a^3.na^{n-1}}=\frac{-2y^2}{b^3.nb^{n-1}} \Rightarrow \frac{x^2}{a^{n+2}}=\frac{y^2}{b^{n+2}}$$

$$\Rightarrow \frac{x^2/a^2}{a^n}=\frac{y^2/b^2}{b^n}=\frac{(x^2/a^2)+(y^2/b^2)}{a^n+b^n}=\frac{1}{c^n};\ \text{from (1) and (2)}$$

$$\Rightarrow \frac{x^2}{a^{n+2}}=\frac{1}{c^n} \text{ and } \frac{y^2}{b^{n+2}}=\frac{2}{c^n}$$

$$\Rightarrow a^{n+2}=c^nx^2,\ b^{b+2}=c^ny^2$$

$$\Rightarrow a^n=(c^nx^2)^{\frac{n}{n+2}};\ b^n=(c^ny^2)^{\frac{n}{n+2}}.$$

Adding, we get

$$a^n+b^n=c^{\frac{n^2}{n+2}}x^{\frac{2n}{n+2}}+c^{\frac{n^2}{n+2}}y^{\frac{2n}{n+2}}$$

$$\Rightarrow c^n=c^{\frac{n^2}{n+2}}\left\{x^{\frac{2n}{n+2}}+y^{\frac{2n}{n+2}}\right\},$$

$$\Rightarrow x^{\frac{2n}{n+2}}+y^{\frac{2n}{n+2}}=c^{n-\frac{n^2}{n+2}}=c^{\frac{2n}{n+2}},$$

Cor. For $n=1$ the reqd. envelope is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$. **(Gorakhpur 90)**

Example 36

1. Find the envelopes of the following families of st. lines.

(i) $y=mx+\sqrt{a^2m^2+b^2}$; parameter m

(ii) $y=m^2x+\frac{1}{m^2}$; parameter m

(iii) $x\cos\alpha+y\sin\alpha=4$; parameter α.

(iv) $y\cos\theta-x\sin\theta=a-a\sin\theta\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$; parameter θ

(v) $y=mx+am^p$; parameter m

2. Find the envelopes of the following families of curves

(i) $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$; parameter α

(ii) $x = a \sin(\theta - \alpha)$, $y = b\cos\theta$; parameter α

(iii) $ay^2 = \alpha^2(x - \alpha)$; parameter α

(iv) $(x - \alpha)^2 + y^2 = 4a$; parameter α.

3. Find the envelope of the family of st. lines $\frac{x}{a} + \frac{y}{b} = 1$ where a, b are connected by the relation.

(i) $a + b = c$ (ii) $a^2 + b^2 = c^2$

4. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters a, b are connected by the relation

(i) $ab = c^2$ (ii) $a^2 + b^2 = c^2$

5. Find the envelope of the curves $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ where the parameters a and b are connected by the relation $a^p + b^p = c^p$.

6. Show that the family of circles $(x - a)^2 + y^2 = a^2$ has no envelope.

7. From any point P on the parabola $y^2 = 4ax$, perpendiculars PM, PN are drawn to the co-ordinate axes, find the envelope of the line MN.

8. From any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, perpendiculars are drawn to the axes and their feet are joined. Show that the st line thus formed always touches the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

9. Show that the envelope of the st. line joining the extremities of a pair of semi-conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$$

10. A circle moves with its centre on the parabola $y^2 = 4ax$ and always passes through the vertex of the parabola. Show that the envelope of the circles is the curve $x^3 + y^2(x + 2a) = 0$.

11. Show that the envelope of the circles whose centre lies on the rectangular hyperbola $x^2 - y^2 = a^2$ and which pass through the origin is the curve $r^2 = 4a^2\cos 2\theta$.

12. Considering the evolute of a curve as the envelope of its normals , find the evolute of the parabola $y^2 = 4ax$.

13. Prove that the equation of the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ may be written in the form $x\sin\phi - y\cos\phi + a\cos 2\phi = 0$ and find the envelope of the normals.

14. Find the envelope of circles described on the radii vectors of $y^2 = 4ax$ as diameter.

15. Circles are described on the double ordinates of the parabola $y^2 = 4ax$ as diameters ; prove that the envelope is $y^2 = 4a(x + a)$.

16. Circles are described on the double ordinates of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameters; prove that the envelope is $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$.

17. Find the envelope of the circles described on the radii vectors of the curve $r^2 = a^2\cos 2\theta$ as diameter.

18. Find the envelope of lines drawn through the extremities of and at right angles to the radii vectors of the curves.

(i) $r = ae^{m\theta}$ (ii) $r = ae^{\theta\cot\alpha}$

19. Show that the radius of curvature of the envelope of the line

$$x\cos\theta + y\sin\theta = f(\theta) \text{ is } f(\theta) + f''(\theta).$$

20. Given that the astroid $x^{2/3} + y^{2/3} = c^{2/3}$ is the envelope of $\frac{x}{a} + \frac{y}{b} = 1$, prove that a and b are connected by $a^2 + b^2 = c^2$. If further, the astroid is the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $a + b = c$.

ANSWERS

1. (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $y^2 = 4x$ (iii) $x^2 + y^2 = 16$ (iv) $y = a\cos b\frac{x}{a}$

(v) $(p-1)^{p-1}x^p + p^p a y^{p-1} = 0$

2. (i) $(x + y + 1)^2 = 2(x^2 + y^2)$ (ii) $x = \pm a$

(iii) $ay^2 = \frac{4}{27}x^3$ (iv) $y^2 - 4x - 4 = 0$

3. (i) $\sqrt{x} + \sqrt{y} = \sqrt{c}$ (ii) $x^{2/3} + y^{2/3} = c^{2/3}$

4. (i) $2xy = c^2$ (ii) $x \pm y \pm c = 0$

5. $x^{mp/(m+p)} + y^{mp/(m+p)} = c^{mp/(m+p)}$

7. $y^2 = -16ax$ **12.** $27ay^2 = 4(x - 2a)^3$

13. $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$ **14.** $ay^2 + x(x^2 + y^2) = 0$

17. rectangular hyperbola

18. (i) an equiangular spiral

(ii) $r \sin\alpha = ae^{(\alpha - \pi/2)\cot\alpha}.e^{\theta \cot\alpha}$

11

MAXIMA AND MINIMA OF A FUNCTION OF TWO VARIABLES

11.1. In the previous class, we have learnt about the maxima and minima of a function of one variable only. Here in this chapter, we shall define the maxima and minima of a function of two variables x and y. The treatment is quite similar. Before we start, we would like to revise the definition and criteria of a function to have a maximum or a minimum in its domain.

***Definition*—Maximum value**—If a function $f(x)$ be such that its value $f(c)$ at the point $x = c$ is greater than the ordinate $f(x)$ for every point x lying in the small neighbourhood of $x = c$ i.e. $|x - c| < \delta$ i.e. in the interval $c - \delta < x < c + \delta$, then we say that the function $f(x)$ possesses a maximum value at $x = c$.

Minimum value—In the similar manner, if the value of $f(c)$ is less than the ordinate $f(x)$ for every point x lying in the interval $c - \delta < x < c + \delta$, then we say that the function possesses a mimimum value at $x = c$.

We also know that if a function $f(x)$ has a max. or min. value at $x = c$, then $f'(c) = 0$ necessarily. And in addition if $f''(c) < 0$, then the function has a max. value at $x = c$ and if $f''(c) > 0$, then the function has a min. value.

Now we proceed to define and discuss the maxima and minima of a function of two variables.

Let $f(x, y)$ be a function of two variables x and y and let (a, b) be a point in its domain of definition.

***Definition*—Maximum**—The function $f(x, y)$ is said to have maximum value at a point (a, b), provided $f(x, y) < f(a, b)$ i.e. $f(x, y) - f(a, b) < 0$ for every point (x, y) in the neighbourhood of (a, b).

In other words, $f(x, y)$ is said to possess a maximum value at (a, b) provided $f(a + h, b + k) - f(a, b) < 0$ where $(a + h, b + k)$ lies in the neighbourhood of (a, b).

Minimum—The function $f(x, y)$ is said to have a minimum value at a point (a, b) provided $f(x, y) > f(a, b)$ i.e. $f(x, y) - f(a, b) > 0$ for every point (x, y) in the neighbourhood of (a, b).

In other words $f(x, y)$ is said to possess a minimum value at (a, b) provided $f(a + h, b + k) - f(a, b) > 0$, where $(a + h, b + k)$ lies in the neighbourhood of (a, b).

The maximum or minimum value of a function is called its **extreme value.**

For determining the criteria for the max. or min. values of a function of two variables, we shall require the application of Taylor's Theorem. Hence we would like to give a formal proof of Taylor's Theorem first (The elementary proof of Taylor's Theorem has already been given in chapter 3 page 150).

11.2. Taylor's Theorem for a function of two variables.

If $f(x, y)$ possesses continuous partial derivatives of the nth order in any neighbourhood of a point (a, b) and if $(a + h, b + k)$ be any point in this neighbourhood, then

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial k} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k)$$

where $0 < \theta < 1$.

We shall require the following lemma to prove the above theorem.

Lemma :

Let $u = f(x, y)$, and let $x = a + ht$, $y = b + kt$,

so that u is a function of t.

We know from the chapter on partial differentiation that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k = h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y}.$$

Thus, we have,

$$\frac{du}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) u.$$

This equality shows that the operators $\frac{d}{at}$ and $h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$ are equivalent

Further, we obtain

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)u$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)u.$$

If now we agree to write

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) \equiv \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2$$

then we shall have

$$\frac{d^2u}{dt^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 u.$$

Continuing in this manner (or by method of induction) it can be shown

$$\frac{d^nu}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n u.$$

Now, we come to the proof of Taylor's Theorem.

We write $x = a + ht$, $y = b + kt$ so that

$$f(x, y) = f(a + ht, b + kt) = g(t),$$

where $g(t)$ is a function of the single variable t.

Now, we apply Maclaurin's theorem to the function $g(t)$.

Accordingly, we have

$$g(t) = g(0) + tg'(0) + \frac{t^2}{2!}g''(0) + \ldots + \frac{t^{n-1}}{(n-1)!}g^{n-1}(0) + \frac{t^n}{n!}g^n(\theta t) \qquad \ldots(1)$$

where $0 < \theta < 1$.

Putting $t = 1$, we get

$$g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \ldots + \frac{1}{(n-1)!}g^{n-1}(0) + \frac{1}{n!}g^n(\theta).$$

Now, $g(1) = f(a, + h, b + k)$, $g(0) = f(a, b)$.

Also, from the lemma,

$$g^n(t) = \frac{d^n}{dt^n}g(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n g(t).$$

Therefore,

$$g'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$g''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

.....

$$g^{n-1}(0) = \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$g^n(\theta) = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k).$$

Substituting these values in (1), we get the Taylor's Theorem as stated.

11.3. Criteria for determination of maxima or minima of a function of two variables.

The necessary condition :

Let $f(x, y)$ be a function of two variables x and y.

If $f(a, b)$ is an extreme value of the function $f(x, y)$, then clearly the functions $f(x, b)$ of one variable x for $x = a$ and $f(a, y)$ of one variable y for $y = b$ must have the same extreme value. Hence from the knowledge of a function of one variable the necessary conditions for these functions to have extreme value at (a, b) are respectively

$$\frac{\partial}{\partial x} f(a, b) = 0 \text{ and } \frac{\partial}{\partial y} f(a, b) = 0.$$

***Definition*—Stationary point**—The point (x, y) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ is called a stationary point for the function $f(x, y)$.

Sufficient conditions :

Let $f(x, y)$ be a function of two variables x and y.

Let us denote $\frac{\partial^2 f}{\partial x^2}$ at (a, b) by r,

$\frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$ at (a, b) by s;

and $\frac{\partial^2 f}{\partial y^2}$ at (a, b) by t.

We are going to show that :

If $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ at (a, b) and higher partial derivatives exist in the neighbourhood of (a, b), then $f(x, y)$ will have

(i) a maximum at (a, b) if $r < 0$ and $rt - s^2 > 0$,

(ii) a minimum at (a, b) if $r > 0$ and $rt - s^2 > 0$;

(iii) neither a maximum nor a minimum at (a, b) if $rt - s^2 < 0$ and $r \neq 0$;

(iv) if $rt - s^2 = 0$, the case is doubtful and needs further consideration.

It may be noted that if $rt - s^2 > 0$, then $r \neq 0$. (for in that case—s^2 becomes > 0, which is not true).

By Taylor's Theorem (with remainder after three terms), we obtain

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a + \theta h, b + \theta k) \text{ where } 0 < \theta < 1.$$

$$= f(a, b) + 0 + \frac{1}{2!}\left\{h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right\} f(a, b) + \rho$$

since $\frac{\partial}{\partial x} f(a, b) = 0$, $\frac{\partial}{\partial y} f(a, b) = 0$

and where ρ is of third degree in h and k.

$\Rightarrow$ $f(a + h, b + k) - f(a, b)$

$$= \frac{1}{2!}\left\{h^2\frac{\partial^2}{\partial x^2} f(a, b) + 2hk\frac{\partial^2}{\partial x \partial y} f(a, b) + k^2\frac{\partial^2}{\partial y^2} f(a, b)\right\} + \rho$$

$$= \frac{1}{2}\{h^2 r + 2hks + k^2 t\} + \rho.$$

Obviously for sufficiently small values of h, k, the sign of $f(a + h, b + k) - f(a, b)$ is determined by $\frac{1}{2}\{h^2 r + 2hks + k^2 t\}$ i.e. by $h^2 r + 2hks + k^2 t$. In other words, $h^2 r + 2hks + k^2 t$ is the dominant expression.

Now, ignoring higher powers of h and k and hence ρ, we have approximately

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}(h^2 r + 2hks + k^2 t)$$

$$= \frac{1}{2r}(h^2 r^2 + 2hkrs + k^2 rt) = \frac{1}{2r}\{(hr + ks)^2 + (rt - s^2)k^2\}.$$

Now, we discuss the sign of $f(a + h, b + k) - f(a, b)$ vis—a—vis the sign of $\frac{1}{2r}\{(hr + ks)^2 + (rt - s^2)k^2\}$.

Accordingly, we discuss the following three cases.

Case 1. Let $rt - s^2 > 0$.

In this case, neither r nor t can be zero.

Since $rt - s^2 > 0$, we find that $(rh + ks)^2 + (rt - s^2)k^2$ is always positive, except when $rh + sk = 0$, $k = 0$, i.e. when $h = 0$, $k = 0$.

Thus, we see that $rh^2 + 2shk + tk^2$ always retains the same sign as that of r.

Thus $f(a + h, b + k) - f(a, b) > 0$ if $r > 0$

and $f(a + h, b + k) - f(a, b) < 0$ if $r < 0$.

That is, $f(a, b)$ will be a minimum if $r > 0$,

and $f(a, b)$ will be a maximum if $r < 0$.

Hence, if $rt - s^2 > 0$, and if

(i) $r < 0$, $f(x, y)$ will be a max. at (a, b)

(ii) $r > 0$, $f(x, y)$ will be a min. at (a, b).

Case 2. Let $rt - s^2 < 0$.

Firstly, we suppose that $r \neq 0$. We write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}\{(rh + sk)^2 + (rt - s^2)k^2\}.$$

Since $rt - s^2 < 0$, we see that this expression sometimes takes up positive signs and sometimes takes up negative signs except when $k = 0$ and $rh + sk = 0$.

Thus in this case $f(a, b)$ is not an extreme value.

The proof is similar when $t \neq 0$.

In case, $r = 0$ as well as $t = 0$, we have,

$$rh^2 + 2shk + tk^2 = 2shk$$

so that the expression assumes values with different signs and consequently $f(a, b)$ is not an extreme value.

Case 3. Let $rt - s^2 = 0$.

Suppose that $r \neq 0$. Then we have

$$rh^2 + 2shk + tk^2 = \frac{1}{r}(rh + sk)^2 + (rt - s^2)k^2\} = \frac{(rh + sk)^2}{r}.$$

Here the expression $rh^2 + 2shk + tk^2$ becomes zero when $rh + sk = 0$, so that the nature of the sign of $f(a + h, b + k) - f(a, b)$ depends upon the consideration of ρ.

The case is, therefore doubtful.

If now $r = 0$, then because of the condition $rt - s^2 = 0$, we must have $s = 0$. Therefore $rh^2 + 2shk + tk^2 = tk^2$ so that the expression is zero when $k = 0$ whatever h may be.

The case is again doubtful.

11.4. Summary

To sum up

1. **If $rt - s^2 > 0$ and**
   if $r < 0$, then $f(x, y)$ will be a max. at (a, b);
   if $r > 0$, then $f(x, y)$ will be a min. at (a, b).
2. **If $rt - s^2 < 0$, then $f(a, b)$ is not an extreme value**
3. **if $rt - s^2 = 0$, then the case is doubtful and we require to consider the higher orders of the differential operator $h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$.**

Worked out Examples

EXAMPLE 1. *Find the maxima and minima of the function*

$$x^3 + y^3 - 3x - 12y + 10.$$

SOLUTION : Let $u = x^3 + y^3 - 3x - 12y + 10$.

Then $$\frac{\partial u}{\partial x} = 3x^2 - 3 = 3(x^2 - 1)$$

$$\frac{\partial u}{\partial y} = 3y^2 - 12 = 3(y^2 - 4).$$

For max. or min. values of u, we have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

$\therefore$ $$x^2 - 1 = 0 \qquad \ldots(1)$$

and $$y^2 - 4 = 0. \qquad \ldots(2)$$

Solving (1) and (2), we get $x = \pm 1$, $y = \pm 2$.

Hence the stationary points are

$$(1, 2), (1, -2), (-1, 2), (-1, -2).$$

Now $$r = \frac{\partial^2 u}{\partial x^2} = 6x, \quad s = \frac{\partial^2 u}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

(i) At the point $(1, 2)$, $r = 6$, $s = 0$, $t = 12$.

$\therefore$ $$rt - s^2 = 6 \times 12 - 0 = 72 (> 0).$$

Thus $r > 0$ and $rt - s^2 > 0$.

Hence u has a minimum at $(1, 2)$.

(ii) At the point $(1, -2)$, $r = 6$, $s = 0$, $t = -12$.

$\therefore$ $rt - s^2 = -72(< 0)$.

Thus $r \neq 0$ and $rt - s^2 < 0$.

Hence u has neither a maximum nor a minimum at $(1, -2)$.

(iii) At the point $(-1, 2)$, $r = -6$, $s = 0$, $t = 12$.

$\therefore$ $rt - s^2 = -72(< 0)$.

Thus $r \neq 0$ and $rt - s^2 < 0$.

Hence u has neither a maximum nor a minimum at $(-1, 2)$.

(iv) At the point $(-1, -2)$, $r = -6$, $s = 0$, $t = -12$.

$\therefore$ $rt - s^2 = 72(> 0)$.

Thus $r < 0$ and $rt - s^2 > 0$.

Hence u has a maximum at $(-1, -2)$.

EXAMPLE 2. *Find the maximum or minimum values of*

$$x^3 + y^3 - 3axy; \quad x \neq 0, y \neq 0.$$ **(P.U. 71H; B.U. 71H)**

SOLUTION : Let $u = x^3 + y^3 - 3axy$.

Then $$\frac{\partial u}{\partial x} = 3x^2 - 3ay \qquad \ldots(1)$$

and $$\frac{\partial u}{\partial y} = 3y^2 - 3ax. \qquad \ldots(2)$$

For max. or min. values of u, we have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

$\therefore$ $3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$.

That is, $$x^2 = ay \qquad \ldots(3)$$

and $$y^2 = ax. \qquad \ldots(4)$$

Squaring (3), we get

$$x^4 = a^2y^2 = a^2.ax; \text{ from (4)}$$

$\Rightarrow$ $x(x^3 - a^3) = 0 \quad \therefore \quad x = a$; since $x \neq 0$.

Hence $y = a$.

Thus the stationary point is (a, a) at which the function may have the maximum or minimum value.

Now $$r = \frac{\partial^2 u}{\partial x^2} = 6x, \quad s = \frac{\partial^2 u}{\partial x \partial y} = -3a, \quad r = \frac{\partial^2 u}{\partial y^2} = 6y$$

$\therefore$ At (a, a), $r = 6a$, $s = -3a$, $t = 6a$.

$\therefore \quad rt - s^2 = (6a)(6a) - (3a)^2 = 36a^2 - 9a^2 = 27a^2 (> 0).$

Thus $r = 6a$ and $rt - s^2 > 0$.

Hence if $a < 0$, then $r < 0$ and $rt - s^2 > 0$ and consequently the function has a max. value at (a, a).

On the other hand, if $a > 0$, then $r > 0$ and also $rt - s^2 > 0$ and hence the function has a min. value at (a, a).

EXAMPLE 3. *Find the maximum or minimum value of* $x^3y^2(a - x - y), \quad x \neq 0, \quad y \neq 0, \; x + y \neq a.$

SOLUTION : Let $u = x^3y^2(a - x - y)$.

Then $$\frac{\partial u}{\partial x} = 3x^2y^2(a - x - y) - x^3y^2 \qquad ...(1)$$

and $$\frac{\partial u}{\partial y} = 2x^3y(a - x - y) - x^3y^2. \qquad ...(2)$$

For max. or min values of u, we must have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

Hence $$3x^2y^2(a - x - y) = x^3y^2 \qquad ...(3)$$

$$2x^3y(a - x - y) = x^3y^2. \qquad ...(4)$$

From (3) and (4), we get

$$3x^2y^2(a - x - y) = 2x^3y(a - x - y)$$

$$\Rightarrow \quad 3x^2y^2(a - x - y) - 2x^3y(a - x - y) = 0$$

$$\Rightarrow \quad x^2y(a - x - y)(3y - 2x) = 0$$

$$\Rightarrow \quad 3y - 2x = 0; \text{ since } x \neq 0, \; y \neq 0 \text{ and } x + y \neq a$$

$$\therefore \quad y = \frac{2}{3}x.$$

From (3), we get

$$3x^2 \cdot \frac{4}{9}x^2\left(a - x - \frac{2}{3}x\right) = x^3 \cdot \frac{4}{9}x^2$$

$$\Rightarrow \quad \frac{4}{3}x^4\left(a - \frac{5x}{3}\right) = \frac{4}{9}x^5$$

$$\Rightarrow \quad a - \frac{5x}{3} = \frac{x}{3} \Rightarrow \frac{5x}{3} + \frac{x}{3} = a.$$

$$\Rightarrow \quad 2x = a \quad \therefore \quad x = \frac{a}{2}.$$

$$\therefore \quad y = \frac{2}{3}x = \frac{2}{3} \cdot \frac{a}{2} = \frac{a}{3}.$$

Now $r = \dfrac{\partial^2 u}{\partial x^2} = 3y^2\{x^2(-1) + (a - x - y).2x\} - 3x^2y^2$

$= -3x^2y^2 + 6xy^2(a - x - y) - 3x^2y^2$

$= 6xy^2(a - x - y) - 6x^2y^2,$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 3x^2\{y^2(-1) + (a - x - y).2y\} - 2x^3y$

$= 6x^2y(a - x - y) - 3x^2y^2 - 2x^3y.$

$t = \dfrac{\partial^2 u}{\partial y^2} = 2x^3\{y(-1) + (a - x - y)(1)\} - x^3.2y$

$= 2x^3(a - x - y) - 3x^3y.$

Hence at $\left(\dfrac{a}{2}, \dfrac{a}{3}\right)$, we have

$$r = 6 \cdot \frac{a}{2} \cdot \frac{a^2}{9}\left(a - \frac{a}{2} - \frac{a}{3}\right) - 6 \cdot \frac{a^2}{4} \cdot \frac{a^2}{9}$$

$$= \frac{a^3}{3}\left(\frac{a}{6}\right) - \frac{a^4}{6} = \frac{a^4}{6}\left(\frac{1}{3} - 1\right) = -\frac{a^4}{6} \cdot \frac{2}{3}$$

$$= -\frac{1}{9}a^4 (< 0).$$

$$s = 6 \cdot \frac{a^2}{4} \cdot \frac{a}{3}\left(a - \frac{a}{2} - \frac{a}{3}\right) - 3\frac{a^2}{4} \cdot \frac{a^2}{9} - 2 \cdot \frac{a^3}{8} \cdot \frac{a}{3}$$

$$= \frac{a^3}{2} \cdot \left(\frac{a}{6}\right) - \frac{a^4}{12} - \frac{a^4}{12}$$

$$= \frac{a^4}{12} - \frac{a^4}{12} - \frac{a^4}{12} = -\frac{a^4}{12},$$

$$t = 2 \cdot \frac{a^3}{8}\left(a - \frac{a}{2} - \frac{a}{3}\right) - 4 \cdot \frac{a^3}{8} \cdot \frac{a}{3}$$

$$= \frac{a^3}{4}\left(\frac{a}{6}\right) - \frac{a^4}{6} = \frac{a^4}{6}\left(\frac{1}{4} - 1\right) = \frac{a^4}{6}\left(-\frac{3}{4}\right) = -\frac{a^4}{8}.$$

Now $rt - s^2 = \left(-\dfrac{a^4}{9}\right)\left(-\dfrac{a^4}{8}\right) - \left(-\dfrac{a^4}{12}\right)^2$

$= \dfrac{a^8}{72} - \dfrac{a^8}{144} = \dfrac{a^8}{72}\left(1 - \dfrac{1}{2}\right) = \dfrac{a^8}{144} (> 0).$

Thus $r < 0$ and $rt - s^2 > 0$.

Hence u is maximum at $\left(\dfrac{a}{2}, \dfrac{a}{3}\right)$.

The maximum value of the given function is, therefore

$$= \left(\frac{a}{2}\right)^3 \left(\frac{a}{3}\right)^2 \left(a - \frac{a}{2} - \frac{a}{3}\right)$$

$$= \frac{a^3}{8} \cdot \frac{a^2}{9} \cdot \left(\frac{a}{6}\right) = \frac{a^6}{432}.$$

EXAMPLE 4. *Find the extreme value of* $2(x - y)^2 - x^4 - y^4$.

SOLUTION : Let us write $f(x, y) = 2(x - y)^2 - x^4 - y^4$.

Then $$\frac{\partial f}{\partial x} = 4(x - y) - 4x^3 \qquad ...(1)$$

and $$\frac{\partial f}{\partial y} = -4(x - y) - 4y^3. \qquad ...(2)$$

For max. or min. value of u, we must have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Therefore $4(x - y) - 4x^3 = 0 \Rightarrow (x - y) - x^3 = 0.$ ...(3)

and $-4(x - y) - 4y^3 = 0 \Rightarrow (x - y) + y^3 = 0.$ ...(4)

We need to solve for x and y from equations (3), (4).

From (3) and (4), we have

$$x^3 + y^3 = 0 \text{ i.e. } (x + y)(x^2 - xy + y^2) = 0.$$

Hence either $x + y = 0$ or $x^2 - xy + y^2 = 0$.

Thus we have to solve two sets of equations.

$$\left.\begin{aligned} x + y &= 0 \\ x - y - x^3 &= 0 \end{aligned}\right] \qquad ...(5)$$

and

$$\left.\begin{aligned} x^2 - xy + y^2 &= 0 \\ x - y - x^3 &= 0 \end{aligned}\right] \qquad ...(6)$$

From (5), we have (on putting $y = -x$),

$$x + x - x^3 = 0 \Rightarrow 2x - x^3 = 0 \Rightarrow x(2 - x^2) = 0.$$

$\therefore$ $x = 0$ or $x = \pm \sqrt{2}$.

Hence $y = 0$ or $y = \mp \sqrt{2}$.

Hence the solutions obtained from (5) are

$$(0, 0), \ (\sqrt{2}, -\sqrt{2}) \text{ and } (-\sqrt{2}, \sqrt{2}).$$

Again, from (6), we have

$$x^2 - xy + y^2 = 0$$

and $$x^2 - xy - x^4 = 0,$$

multiplying the second equation of (6) by x.

On subtraction, $y^2 + x^4 = 0$ which $\Rightarrow$ $x = 0$, $y = 0$ for the real solution of x and y.

Hence, finally for the max. or min. value of f, the stationary points are

$$(0, 0), (\sqrt{2}, -\sqrt{2}) \text{ and } (-\sqrt{2}, \sqrt{2}).$$

Now, from (1) and (2),

$$r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2; \; s = \frac{\partial^2 f}{\partial x \partial y} = -4, \; t = \frac{\partial^2 f}{\partial y^2} = 4 - 12y^2.$$

Therefore, at (0, 0), $r = 4$, $s = -4$, $t = 4$.

$\therefore$ $rt - s^2 = 16 - 16 = 0.$

Hence (0, 0) is a doubtful point.

Again at $(\sqrt{2}, -\sqrt{2})$,

$$r = 4 - 24 = -20, \; s = -4, \; t = 4 - 24 = -20.$$

$\therefore$ $rt - s^2 = 400 - 16 = +ve.$

Since $r < 0$ and $rt - s^2 > 0$, therefore the point $(\sqrt{2}, -\sqrt{2})$ gives the maximum value of $f(x, y)$.

Again, at $(-\sqrt{2}, \sqrt{2})$,

$$r = 4 - 24 = -20, \; s = -4, \; t = 4 - 24 = -20.$$

$$rt - s^2 = 400 - 16 = + ve.$$

Here again, since $r < 0$ and $rt - s^2 > 0$, therefore the point $(-\sqrt{2}, \sqrt{2})$ gives the maximum value of $f(x, y)$.

Hence the given function attains the maximum value at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

Note : The case that arises when $x = 0$ 0 can be disposed of by an elementary consideration as follows :

We have, $f(0, 0) = 0$.

For points $(x, 0)$ along the x-axis,

$$f(x, 0) = 2x^2 - x^4 = x^2(2 - x^2),$$

which is positive in the neighbourhood of the origin.

Again, for points along the line $y = x$,

$$f(x, y) = f(x, x) = -2x^4 (< 0).$$

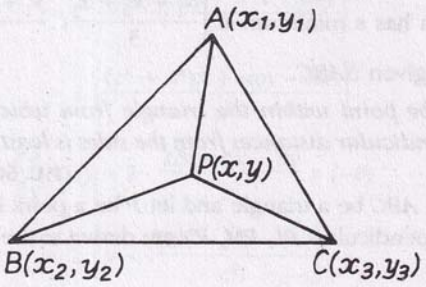
Thus we see that every neighbourhood of the origin (0, 0) there are points where function assumes positive values i.e. $> f(0, 0)$ and there are points where the function assumes negative values i.e. $< f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value.

EXAMPLE 5. *Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.* **(P.U. 57H; Bhag. 62H, 70H)**

SOLUTION : Let ABC be a triangle whose vertices A, B, C are respectively (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Let the required point be $P(x, y)$.



We are required to find the point P inside the ΔABC such that $PA^2 + PB^2 + PC^2$ is minimum.

Let $$u = PA^2 + PB^2 + PC^2$$

$$= \{(x - x_1)^2 + (y - y_1)^2\} + \{(x - x_2)^2 + (y - y_2)^2\} + \{(x - x_3)^2 + (y - y_3)^2\}$$

$$= \sum_{i=1}^{3} \{(x - x_i)^2 + (y - y_i)^2\}.$$

For max. or min. we must have $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0$.

Now, $\dfrac{\partial u}{\partial x} = \sum_{i=1}^{3} 2(x - x_i)$ and $\dfrac{\partial u}{\partial y} = \sum_{i=1}^{3} 2(y - y_i)$.

Hence $\sum_{i=1}^{3} (x - x_i) = 0$ and $\sum_{i=1}^{3} (y - y_i) = 0$.

This $\Rightarrow$ $3x - (x_1 + x_2 + x_3) = 0$ and $3y - (y_1 + y_2 + y_3) = 0$

i.e. $x = \dfrac{x_1 + x_2 + x_3}{3}$ and $y = \dfrac{y_1 + y_2 + y_3}{3}$.

Now, we proceed to show that with this value of (x, y), u is minimum.

We have,

$$r = \frac{\partial^2 u}{\partial x^2} = 2.3 = 6, \quad s = \frac{\partial^2 u}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 u}{\partial y^2} = 2.3 = 6$$

$$\therefore \qquad rt - s^2 = 36 - 0 = 36 \ (> 0).$$

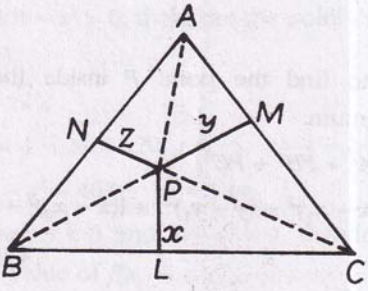
Thus $r > 0$ and $rt - s^2 > 0$.

$\therefore$ The function has a minimum at $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ which is the centroid of the given ΔABC.

EXAMPLE 6. *Find the point within the triangle from which the sum of the squares of its perpendicular distances from the sides is least.*

(P.U. 60H; B.U. 59H, 62

SOLUTION : Let ABC be a triangle and let P be a point inside the triangle from which the perpendiculars PL, PM, PN are drawn to the sides BC, CA and AB respectively.



Let $\qquad PL = x, \quad PM = y$ and $PN = z$.

Let $\qquad u = x^2 + y^2 + z^2.$ ...(1)

We are required to find the min. value of u.

Now $\Delta ABC = \Delta PBC + \Delta PCA + \Delta PAB$

$$= \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz$$

i.e. $\qquad \Delta = \frac{1}{2}(ax + by + cz)$

i.e. $\qquad ax + by + cz = 2\Delta.$ ...(2)

Thus from (1) and (2), we have

$$u = x^2 + y^2 + \left(\frac{2\Delta - ax - by}{c}\right)^2. \qquad \text{...(3)}$$

Now from (1), we have

$$\frac{\partial u}{\partial x} = 2x + 2 \cdot \frac{(2\Delta - ax - by)}{c^2}(-a)$$

$$= 2\left\{x - \frac{a(2\Delta - ax - by)}{c^2}\right\}$$

$$= 2\left\{\frac{c^2x - 2a\Delta + a^2x + aby}{c^2}\right\}$$

$$= 2\left\{\frac{(c^2 + a^2)x + aby - 2a\Delta}{c^2}\right\}$$

and $$\frac{\partial u}{\partial y} = 2y + 2 \cdot \frac{2\Delta - ax - by}{c^2} \times (-b)$$

$$= 2\left\{y - \frac{b(2\Delta - ax - by)}{c^2}\right\}$$

$$= 2\left\{\frac{c^2y - 2b\Delta + abx + b^2y}{c^2}\right\}$$

$$= 2\left\{\frac{(c^2 + b^2)y + abx - 2b\Delta}{c^2}\right\}$$

For max. or min. value of u, we must have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

Therefore $(c^2 + a^2)x + aby - 2a\Delta = 0$

and $(c^2 + b^2)y + abx - 2b\Delta = 0$

That is, $(c^2 + a^2)x + aby = 2a\Delta$ ...(4)

and $abx + (c^2 + b^2)y = 2b\Delta$ ...(5)

Multiplying (4) by $(c^2 + b^2)$ and (5) by ab; and then subtracting, we get

$$\{(c^2 + a^2)(c^2 + b^2) - a^2b^2\}x = 2a\Delta(c^2 + b^2) - (2b\Delta)(ab)$$

$\Rightarrow$ $\{c^4 + c^2(a^2 + b^2)\}x = 2\Delta(ac^2 + ab^2 - ab^2)$

$\Rightarrow$ $c^2(c^2 + a^2 + b^2)x = 2\Delta ac^2$

$\therefore$ $$x = \frac{2a\Delta}{a^2 + b^2 + c^2}.$$ ...(6)

Similarly $$y = \frac{2b\Delta}{a^2 + b^2 + c^2}$$ ...(7)

and $$z = \frac{2c\Delta}{a^2 + b^2 + c^2}.$$ ...(8)

Thus u will be greatest or least for the point P whose perpendicular distances from the sides are given by (6), (7), (8).

Now we shall determine the max.. or min. value of u. For this,

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{2}{c^2}(c^2 + a^2) = 2\left(1 + \frac{a^2}{c^2}\right)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{2ab}{c^2} \quad \text{and} \quad t = \frac{\partial^2 u}{\partial y^2} = \frac{2}{c^2}(c^2 + b^2) = 2\left(1 + \frac{b^2}{c^2}\right)$$

$$\therefore \quad rt - s^2 = 2\left(1 + \frac{a^2}{c^2}\right) \cdot 2\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left\{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2} + \frac{a^2b^2}{c^4} - \frac{a^2b^2}{c^4}\right\}$$

$$= 4\left\{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right\} = 4\left\{\frac{a^2 + b^2 + c^2}{c^2}\right\} > 0.$$

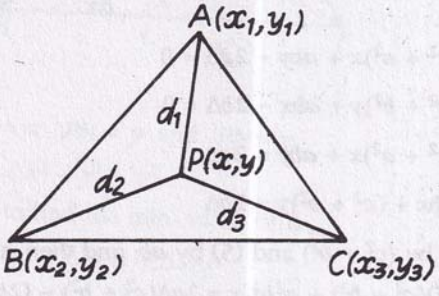
Thus we find that $r > 0$ and $rt - s^2 > 0$.

Hence u is minimum.

Note : For another proof by Lagrange's method, see Ex. 1. worked out on page 420.

EXAMPLE 7. *Find a point within a triangle such that the sum of the distances from the angular points may be a minimum.*

(P.U. 51H, 61H, B.U. 58H, 62H, 64H; R.U. 62H; M.U. 65H)



Let ABC be a triangle and let the co-ordinates of its vertices A, B, C are respectively (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Let $P(x, y)$ be the required point inside the triangle.

Let $u = PA + PB + PC$

$$= \sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

$$+ \sqrt{(x - x_3)^2 + (y - y_3)^2}$$

Then $$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{(x-x_1)^2+(y-y_1)^2}} \times 2(x-x_1)$$

$$+ \frac{1}{2\sqrt{(x-x_2)^2+(y-y_2)^2}} \times 2(x-x_2)$$

$$+ \frac{1}{2\sqrt{(x-x_3)^2+(y-y_3)^2}} \times 2(x-x_3)$$

$$= \frac{x-x_1}{d_1} + \frac{x-x_2}{d_2} + \frac{x-x_3}{d_3} \qquad ...(1)$$

where $d_1 = PA = \sqrt{(x-x_1)^2+(y-y_1)^2}$ etc.

Similarly $$\frac{\partial u}{\partial y} = \frac{y-y_1}{d_1} + \frac{y-y_2}{d_2} + \frac{y-y_3}{d_3} \qquad ...(2)$$

For max. or min. of u we must have $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

Therefore, $$\frac{x-x_1}{d_1} + \frac{x-x_2}{d_2} + \frac{x-x_3}{d_3} = 0 \qquad ...(3)$$

$$\frac{y-y_1}{d_1} + \frac{y-y_2}{d_2} + \frac{y-y_3}{d_3} = 0. \qquad ...(4)$$

Now, we shall simplify the equation (3) and (4) as follows :

Let PA, PB, PC make an angle α, β, γ respectively with the x-axis. Then

$$\tan\alpha = \frac{y-y_1}{x-x_1}, \quad \tan\beta = \frac{y-y_2}{x-x_2}, \quad \tan\gamma = \frac{y-y_3}{x-x_3}.$$

Therefore $$\sin\alpha = \frac{y-y_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} = \frac{y-y_1}{d_1}$$

$$\cos\alpha = \frac{x-x_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} = \frac{x-x_1}{d_1}$$

and similar expressions for $\sin\beta$, $\cos\beta$, $\sin\gamma$ and $\cos\gamma$.

Hence from (3) and (4), we have

$$\cos\alpha + \cos\beta + \cos\gamma = 0 \qquad ...(5)$$

and $$\sin\alpha + \sin\beta + \sin\gamma = 0. \qquad ...(6)$$

That is, $$\cos\alpha + \cos\beta = -\cos\gamma$$

$$\sin\alpha + \sin\beta = -\sin\gamma.$$

Squaring and adding, we get

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 1$$

$$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta) + (\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta) = 1$$

$$\Rightarrow \quad \{2 + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\} = 1$$

$$\Rightarrow \quad \{2 + 2\cos(\alpha - \beta)\} = 1$$

$$\Rightarrow \quad 2\cos(\alpha - \beta) = -1 \Rightarrow \cos(\alpha - \beta) = -\frac{1}{2} = \cos 120^\circ$$

$$\therefore \quad \alpha - \beta = 120^\circ.$$

That is, the angle between PA and PB is 120°.

Similarly it can be shown from equations (5) and (6), that the angle between (PB and PC) and (PC and PA) is each $= 120^\circ$.

Hence the required point P is such that each side of the ΔABC subtends an angle of 120° at P.

Now, we need to show that u is minimum.

From (1) and (2), we have

$$r = \frac{\partial^2 u}{\partial x^2} = \left\{\frac{1}{d_1} + (x - x_1)\cdot\left\{-\frac{1}{d_1^2}\cdot\frac{\partial d_1}{\partial x}\right\} + \ldots\right\}$$

$$= \left[\frac{1}{d_1} - \frac{x - x_1}{d_1^2}\cdot\frac{\partial}{\partial x}\{\sqrt{(x - x_1)^2 + (y - y_1)^2}\} + \ldots\right]$$

$$= \left\{\frac{1}{d_1} - \frac{x - x_1}{d_1^2}\cdot\frac{1}{2\sqrt{(x - x_1)^2 + (y - y_1)^2}} \times 2(x - x_1) + \ldots\right\}$$

$$= \left\{\frac{1}{d_1} - \frac{(x - x_1)^2}{d_1^3}\right\} + \left\{\frac{1}{d_2} - \frac{(x - x_2)^2}{d_2^3}\right\} + \left\{\frac{1}{d_3} - \frac{(x - x_3)^2}{d_3^3}\right\}$$

$$= \frac{1}{d_1}\left\{1 - \left(\frac{x - x_1}{d_1}\right)^2\right\} + \frac{1}{d_2}\left\{1 - \left(\frac{x - x_2}{d_2}\right)^2\right\}$$

$$+ \frac{1}{d_3}\left\{1 - \left(\frac{x - x_3}{d_3}\right)^2\right\}$$

$$= \frac{1}{d_1}(1 - \cos^2\alpha) + \frac{1}{d_2}(1 - \cos^2\beta) + \frac{1}{d_3}(1 - \cos^2\gamma)$$

$$= \frac{\sin^2\alpha}{d_1} + \frac{\sin^2\beta}{d_2} + \frac{\sin^2\gamma}{d_3}.$$

Similarly

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\sum \frac{(x - x_1)(y - y_1)}{d_1^3} = -\sum \frac{\sin\alpha\cos\alpha}{d_1}$$

and $$t = \frac{\partial^2 u}{\partial y^2} = \frac{\cos^2\alpha}{d_1} + \frac{\cos^2\beta}{d_2} + \frac{\cos^2\gamma}{d_3}.$$

Obviously $r > 0$.

Now $$rt - s^2 = \left(\sum \frac{\sin^2\alpha}{d_1}\right)\left(\sum \frac{\cos^2\alpha}{d_1}\right) - \left(\sum \frac{\sin\alpha\cos\alpha}{d_1}\right)^2 \quad ...(7)$$

If we put $$l_1 = \frac{\sin\alpha}{\sqrt{d_1}},\ m_1 = \frac{\sin\beta}{\sqrt{d_2}},\ n_1 = \frac{\sin\gamma}{\sqrt{d_3}}$$

$$l_2 = \frac{\cos\alpha}{\sqrt{d_1}},\ m_2 = \frac{\cos\beta}{\sqrt{d_2}},\ n_2 = \frac{\cos\gamma}{\sqrt{d_3}},$$

then (7) reduces to

$$\begin{aligned} rt - s^2 &= (\Sigma l_1^2)(\Sigma l_2^2) - (\Sigma l_1 l_2)^2 \\ &= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 \\ &= \Sigma(l_1 m_2 - l_2 m_1)^2; \text{ by Lagrange's identity} \\ &> 0. \end{aligned}$$

Thus we find that $r > 0$ and $rt - s^2 > 0$. Hence u is minimum.

EXAMPLE 37

1. Find the maximum or minimum values of the following :

(i) $6x - 4y - x^2 - 2y^2$.

(ii) $x^3 + y^3 - 12x - 3y + 15$.

(iii) $xy(a - x - y)$; $x \neq 0,\ y \neq 0,\ x + y \neq a$.

(iv) $x^3y^2(1 - x - y)$; $x \neq 0,\ y \neq 0, x + y \neq 1$.

(v) $x^3y^2(12 - 3x - 4y)$.

(vi) $x^2y(x + 2y - 4)$.

(vii) $x^4 + y^4 + 4xy - 2x^2 - 2y^2$.

(viii) $xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$.

2. Investigate whether $f(x, y) = x^2 - 3xy^2 + 2y^4$ has an extreme value at $(0, 0)$. **(P.U. 64H)**

3. Find all the stationary points of the function

$$u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$

Also examine whether they are maxima or minima at (4, 0) and (6, 0).

4. Find the maximum value of u where $u = \sin x \sin y \sin(x + y)$.

5. Prove that the function $x^2y^2 - 5x^2 - 8xy - 5y^2$ possesses a maximum value at $x = y = 0$.

6. Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither a maximum nor a minimum at the origin.

7. Show that the maxima and minima of the function

$$u = \frac{ax^2 + 2hxy + by^2 + 2gx + 2fy + c}{a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'}$$

are given by the roots of the equation

$$\begin{vmatrix} a - a'u & h - h'u & g - g'u \\ h - h'u & b - b'u & f - f'u \\ g - g'u & f - f'u & c - c'u \end{vmatrix} = 0. \qquad \textbf{(B.U. 63H, M.U. 66H)}$$

[**Hint :** From the given equation, we have

$u(a'x^2 + 2h'xy + \ldots + c') = ax^2 + 2hxy + \ldots + c.$ ...(1)

Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ and put them equal to zero.

We obtain

$u(a'x + h'y + g') = ax + hy + g$ ...(2)

$u(h'x + b'y + f') = hx + by + f$ ...(3)

Multiplying (2) by x and (3) by y and adding, we get

$u(a'x^2 + 2h'xy + b'y^2 + g'x + f'y) = ax^2 + 2hxy + by^2 + gx + fy$

$\Rightarrow \quad u(g'x + f'y + c') = gx + fy + c$ ...(4) because of (1)

Now, from (2), (3) and (4) the result follows.

ANSWERS

1. (i) max. at (3, –1) (ii) min. at (2, 1); max. at (–2, 1)

(iii) extreme value at $\left(\frac{a}{3}, \frac{a}{3}\right)$; max. if $a > 0$ and min. if $a < 0$.

(iv) max. at $\left(\frac{1}{2}, \frac{1}{3}\right)$ (v) max. at (2, 1). (vi) min. at $\left(2, \frac{1}{2}\right)$.

(vii) min. at $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ (viii) min. at (a, a).

2. $f(x, y)$ does not have the extreme value at (0, 0).

3. (4, 0), (6, 0), (5, 1), (5, –1); max. at (4, 0) and min. at (6, 0).

4. max of u at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$; max. value $= \frac{3\sqrt{3}}{8}$.

11.5 Maxima and Minima under given conditions or constraints.

So far we concerned ourselves with the extreme values of functions assuming that variables of the given function can take any values. However, we come across problems in Mathematics in which there is always a constraint on the variables and as such the variables are not independent. In such circumstances a new technique is used which is called Lagrange's multiplier method.

At the outset, we shall illustrate the concept of undetermined multiplier by taking a function of 3 variables.

To find extreme values of the function $f(x, y, z)$ under the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$.

Let $u = f(x, y, z)$ ...(1)

where $\phi_1(x, y, z) = 0$ ...(2)

and $\phi_2(x, y, z) = 0$ ...(3)

From (1), $du = \dfrac{\partial f}{\partial x}du + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz$

For max. or min. of u, we have $du = 0$,

$\therefore$ $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz = 0$ ...(4)

Also from (2), $d\phi_1 = 0 \Rightarrow \dfrac{\partial \phi_1}{\partial x}dx + \dfrac{\partial \phi_1}{\partial y}dy + \dfrac{\partial \phi_1}{\partial z}dz = 0$ ...(5)

and from (3), $d\phi_2 = 0 \Rightarrow \dfrac{\partial \phi_2}{\partial x}dx + \dfrac{\partial \phi_2}{\partial y}dy + \dfrac{\partial \phi_2}{\partial z}dz = 0$ ...(6)

Now, we multiply (4) by 1, (5) by λ_1 and (6) by λ_2 and then adding them, we get

$$\left\{\frac{\partial f}{\partial x} + \lambda_1\frac{\partial \phi_1}{\partial x} + \lambda_2\frac{\partial \phi_2}{\partial x}\right\}dx + \left\{\frac{\partial f}{\partial y} + \lambda_1\frac{\partial \phi_1}{\partial y} + \lambda_2\frac{\partial \phi_2}{\partial y}\right\}dy + \left\{\frac{\partial f}{\partial z} + \lambda_1\frac{\partial \phi_1}{\partial z} + \lambda_2\frac{\partial \phi_2}{\partial z}\right\}dz = 0 \quad ...(7)$$

Now, if we could put any two of the coeffs of dx, dy and dz to zero, and then solve them, we can calculate λ_1 and λ_2.

So, we choose λ_1 and λ_2 in such a manner that two of the coefficients of dx, dy, dz are equal to zero, say

$$\frac{\partial f}{\partial x} + \lambda_1\frac{\partial \phi_1}{\partial x} + \lambda_2\frac{\partial \phi_2}{\partial x} = 0 \quad ...(8)$$

$$\frac{\partial f}{\partial y} + \lambda_1 \frac{\partial \phi_1}{\partial y} + \lambda_2 \frac{\partial \phi_2}{\partial y} = 0 \qquad ...(9)$$

Then (7) becomes

$$\frac{\partial f}{\partial z} + \lambda_1 \frac{\partial \phi_1}{\partial z} + \lambda_2 \frac{\partial \phi_2}{\partial z} = 0 \qquad ...(10)$$

Thus we have five equations namely (2), (3), (8), (9) and (10) from which we get values of λ_1, λ_2 and the values of x, y, z for which we can get an extreme point for the function $u = f(x, y, z)$.

11.6. Lagrange's Undetermined Multiplier.

Let us consider the general problem of finding the extreme values of a given differentiable function

$$u = f(x_1, x_2, \ldots x_m, u_1, u_2, \ldots u_n) \qquad ...(1)$$

of $m + n$ variables which are connected by n independent differentiable relations:

$$\left.\begin{aligned} F_1(x_1, x_2, \ldots x_m, u_1, u_2, \ldots u_n) &= 0 \\ F_2(x_1, x_2, \ldots x_m, u_1, u_2, \ldots u_n) &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ F_n(x_1, x_2, \ldots x_m, u_1, u_2, \ldots u_n) &= 0 \end{aligned}\right\} \quad ...(2)$$

so that f really depends on m independent variables.

Every system of values of the m independent variables which gives the function of f an extreme value must make $df = 0$.

Also, since the n equations (2) are differentiable, therefore we have the following $(n + 1)$ equations:

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial u_1}du_1 + \ldots &= 0 \\ \frac{\partial F_1}{\partial x_1}dx_1 + \frac{\partial F_1}{\partial x_2}dx_2 + \ldots + \frac{\partial F_1}{\partial u_1}du_1 + \ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \frac{\partial F_n}{\partial x_1}dx_1 + \frac{\partial F_n}{\partial x_2}dx_2 + \ldots + \frac{\partial F_n}{\partial u_1}du_1 + \ldots &= 0 \end{aligned}\right\} \quad ...(3)$$

From these $(n + 1)$ equations, the differentials $du_1, du_2, \ldots du_n$ of the n independent variables can be eliminated and we shall get a result of the form

$$M_1 dx_1 + M_2 dx_2 + \ldots + M_m dx_m = 0.$$

Since the differenitals $dx_1, dx_2, \ldots dx_m$ are independent variables, therefore we get m equations

$$M_1 = 0, \quad M_2 = 0, \ldots M_m = 0.$$

which together with the n equations (2) form a system of $(m + n)$ simultaneous equations between the $(m + n)$ unknowns.

On solving these we get the set of values for which the function f has an extreme value.

However the elimination of the n differentials $du_1, du_2, \ldots du_n$ from the equations (3) above is most conveniently affected in practice by the use of multipliers, a method due to Lagrange. The process is as follows.

Multiply each of the equations (3) by. $1, \lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ respectively which are to be determined. On addition, we obtain

$$\left\{\frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial F_1}{\partial x_1} + \ldots + \lambda_n\frac{\partial F_n}{\partial x_1}\right\}dx_1$$

$$+\left\{\frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial F_1}{\partial x_2} \ldots + \lambda_n\frac{\partial F_n}{\partial x_2}\right\}dx_2$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left\{\frac{\partial f}{\partial x_m} + \lambda_1\frac{\partial F_1}{\partial x_m} + \ldots + \lambda_n\frac{\partial F_n}{\partial x_m}\right\}dx_m$$

$$+\left\{\frac{\partial f}{\partial u_1} + \lambda_1\frac{\partial F_1}{\partial u_1} + \ldots + \lambda_n\frac{\partial F_n}{\partial u_1}\right\}du_1$$

$$+\left\{\frac{\partial f}{\partial u_2} + \lambda_1\frac{\partial F_1}{\partial u_2} + \ldots + \lambda_n\frac{\partial F_n}{\partial u_2}\right\}du_2.$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left\{\frac{\partial f}{\partial u_n} + \lambda_1\frac{\partial F_1}{\partial u_n} + \ldots + \lambda_n\frac{\partial F_n}{\partial u_n}\right\}du_n \qquad \ldots(4)$$

Now assume that the n multipliers have been so chosen that each of the n coefficients of $du_1, du_2, \ldots du_n$ is = 0.

Then in the equation (4), there remain only the differentials of $dx_1, dx_2, \ldots dx_m$.

Since $x_1, x_2, x_3, \ldots x_m$ are independent variables, therefore it follows that the coefficients of $dx_1, dx_2, \ldots dx_m$ are zero.

Hence all the coefficients in (4) vanish and we have the following $(m + n)$ equations.

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial F_1}{\partial x_1} + \ldots + \lambda_n \frac{\partial F_n}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial F_1}{\partial x_2} + \ldots + \lambda_n \frac{\partial F_n}{\partial x_2} = 0$$

.....

$$\frac{\partial f}{\partial x_m} + \lambda_1 \frac{\partial F_1}{\partial x_m} + \ldots + \lambda_n \frac{\partial F_n}{\partial x_m} = 0$$

$$\frac{\partial f}{\partial u_1} + \lambda_1 \frac{\partial F_1}{\partial u_1} + \ldots + \lambda_n \frac{\partial F_n}{\partial u_1} = 0$$

$$\frac{\partial f}{\partial u_2} + \lambda_1 \frac{\partial F_1}{\partial u_2} + \ldots + \lambda_n \frac{\partial F_n}{\partial u_2} = 0$$

.....

$$\frac{\partial f}{\partial u_n} + \lambda_1 \frac{\partial F_1}{\partial u_n} + \ldots + \lambda_n \frac{\partial F_n}{\partial u_n} = 0. \qquad \ldots(5)$$

The systems (2) and (5) contain in all $(2n + m)$ equations and these suffice to determine the values of the n multipliers $\lambda_1, \lambda_2, \ldots \lambda_n$ and the values of the $(m + n)$ variables $x_1, x_2, \ldots x_m, u_1, u_2, \ldots u_n$ for which the function f has an extreme value.

Note : Generally Lagrange's method of undetermined multipliers can only determine points where the function can have a maximum or a minimum. It cannot say definitely whether a given point is a maximum or a minimum. This is also true in the case of a function of three variables. However, when a function of 3 variables is working under a single constraint then the point of max. or min. can definitely be determined by the use of the Art 11 · 6.

This is illustrated in Exs. 1, 2, 5, 6.

EXAMPLE 1. *Find the minimum value of* $x^2 + y^2 + z^2$ *having given*

$$ax + by + cz = p.$$

SOLUTION : Let $u = x^2 + y^2 + z^2$

$$\therefore \qquad du = 2xdx + 2ydy + 2zdz.$$

For extreme value $du = 0$.

$$\therefore \qquad xdx + ydy + zdz = 0. \qquad \ldots(1)$$

Also, given $ax + by + cz - p = 0$

$\therefore \quad adx + bdy + cdz = 0 \quad ...(2)$

Multiplying (1) by 1 and (2) by λ, we get

$$(x + a\lambda)dx + (y + b\lambda)dy + (z + c\lambda)dz = 0$$

$\Rightarrow \quad x + a\lambda = 0 \quad ...(3)$

$y + b\lambda = 0 \quad ...(4)$

$c + c\lambda = 0 \quad ...(5)$

Multiplying (3), (4), (5) by x, y, z respectively, we get

$$x^2 + ax\lambda = 0$$

$$y^2 + by\lambda = 0$$

$$z^2 + cz\lambda = 0$$

Adding $u + \lambda(p) = 0. \quad ...(6)$

Again multiplying (3) by a, (4) by b and (5) by c, we get

$$ax + a^2\lambda = 0$$

$$by + b^2\lambda = 0$$

$$cz + c^2\lambda = 0$$

On adding $p + (a^2 + b^2 + c^2)\lambda = 0, \quad ...(7)$

Now, from (6),

$$\lambda p = -u \quad \therefore \quad \lambda = -u/p$$

and from (7),

$$(a^2 + b^2 + c^2)\lambda = -p \quad \therefore \quad \lambda = \frac{-p}{a^2 + b^2 + c^2}$$

Equating the two values of λ, we get

$$\frac{u}{p} = \frac{p}{a^2 + b^2 + c^2}$$

$$\Rightarrow \quad u = \frac{p^2}{a^2 + b^2 + c^2}.$$

Now, we determine the minimum (or maximum) value of u.

Differentiating $ax + by + cz = p$ partially treating z as a function of x and y, we get

$$a + c\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{a}{c}. \text{ Similarly } \frac{\partial z}{\partial y} = -\frac{b}{c}.$$

Now, given $u = x^2 + y^2 + z^2$,

$$\therefore \quad \frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} = 2x + 2z(-a/c) = 2x - \frac{2a}{c}z$$

$$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} = 2y + 2z(-b/c) = 2y - \frac{2b}{c}z.$$

Hence $$r = \frac{\partial^2 u}{\partial x^2} = 2 - \frac{2a}{c}\frac{\partial z}{\partial x} = 2 - \frac{2a}{c}\cdot(-a/c) = 2 + \frac{2a^2}{c^2}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\frac{2b}{c}\frac{\partial z}{\partial x} = -\frac{2b}{c}(-a/c) = \frac{2ab}{c^2}$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 - \frac{2b}{c}\frac{\partial z}{\partial y} = 2 - \frac{2b}{c}(-b/c) = 2 + \frac{2b^2}{c^2}$$

Here $r > 0$.

Also $$rt - s^2 = \left(2 + \frac{2a^2}{c^2}\right)\left(2 + \frac{2b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left\{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2} + \frac{a^2b^2}{c^4} - \frac{a^2b^2}{c^4}\right\} > 0.$$

Hence we get the min. value of u as above.

EXAMPLE 2. *In a plane triangle ABC, find the maximum value of* $\cos A \cos B \cos C$.

SOLUTION : Let $u = \cos A \cos B \cos C$ ... (1)

We have to find the max. value of u subject to the condition

$$A + B + C = \pi. \qquad ...(2)$$

From (1), Taking log of both sides, we get

$$\log u = \log\cos A + \log\cos B + \log\cos C$$

$$\therefore \quad \frac{1}{u}du = \frac{-\sin A}{\cos A}dA + \frac{-\sin B}{\cos B}dB + \frac{-\sin C}{\cos C}dC$$

$$= -[\tan A dA + \tan B dB + \tan C dC]$$

For max. or min., $du = 0$.

$$\therefore \quad \tan A dA + \tan B dB + \tan C dC = 0 \qquad ...(3)$$

Again, from (2)

$$dA + dB + dC = 0 \qquad ...(4)$$

Now, multiplying (3) by 1 and (4) by λ, we get

$$(\tan A + \lambda)dA + (\tan B + \lambda)dB + (\tan C + \lambda)dC = 0$$

$$\Rightarrow \quad \tan A + \lambda = 0, \ \tan B + \lambda = 0, \ \tan C + \lambda = 0$$

$\Rightarrow \quad \tan A = \tan B = \tan C = -\lambda.$

$\Rightarrow \quad A = B = C.$

But $A + B + C = \pi \quad \therefore \quad A = B = C = \pi/3.$

Now we will determine the maximum (or minimum) value of u.

Since $A + B + C = \pi$, therefore differentiating it partially w.r.t. A treating C as dependent variable, we get

$$1 + \frac{\partial C}{\partial A} = 0 \Rightarrow \frac{\partial C}{\partial A} = -1.$$

Similarly $\dfrac{\partial C}{\partial B} = -1.$

Now $$\frac{\partial u}{\partial A} = \cos B\left\{-\sin A\cos C - \cos A\sin C\frac{\partial C}{\partial A}\right\}$$

$$= \cos B(-\sin A\cos C + \cos A\sin C)$$

$$= \cos B\sin(C - A).$$

$\therefore$ $$r = \frac{\partial^2 u}{\partial A^2} = \cos B\cos(C - A)\left(\frac{\partial C}{\partial A} - 1\right)$$

$$= -2\cos B\cos(C - A).$$

Again $$\frac{\partial u}{\partial B} = \cos A\left\{\cos B(-\sin C)\frac{\partial C}{\partial B} + \cos C(-\sin B)\right\}$$

$$= \cos A(\sin C\cos B - \sin B\cos C)$$

$$= \cos A\sin(C - B).$$

$\therefore$ $$s = \frac{\partial^2 u}{\partial A\partial B} = \cos A\cos(C - B)\frac{\partial C}{\partial A} + \sin(C - B)(-\sin A)$$

$$= -\cos A\cos(C - B) - \sin A\sin(C - B)$$

$$= -\cos(C - B - A) = -\cos(B - C + A)$$

and $$t = \frac{\partial^2 u}{\partial B^2} = \cos A\cos(C - B)\left(\frac{\partial C}{\partial B} - 1\right)$$

$$= -2\cos A\cos(C - B).$$

When $A = B = C = \dfrac{\pi}{3}$, then

$$r = -2\cdot\frac{1}{2}\cdot 1 = -1, \quad s = -\frac{1}{2}, \quad t = -1.$$

$\therefore$ $$rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0 \text{ and } r < 0.$$

Hence the given function is maximum when $A = B = C = \frac{\pi}{3}$.

Hence max. $u = \cos A \cos B \cos C = \cos\frac{\pi}{3} \cdot \cos\frac{\pi}{3} \cdot \cos\frac{\pi}{3}$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

EXAMPLE 3. *Find the maxima and minima of* $x^2 + y^2 + z^2$ *subject to the conditions.* $ax + by + cz = dx + b'y + c'z = 1$. **(P.U. 56H; Bhag. 66H)**

SOLUTION : Let $u = x^2 + y^2 + z^2$.

For maximum or minimum values of u, we have

$$du = 0 \Rightarrow xdx + ydy + zdz = 0 \quad ...(1)$$

$\because$ $ax + by + cz = 1 \quad \therefore \quad adx + bdy + cdz = 0$ ...(2)

Again $\because$ $a'x + b'y + c'z = 1 \quad \therefore \quad a'dx + b'dy + c'dz = 0$ ...(3)

Multiplying (1), (2) and (3) by 1, λ_1 and λ_2 respectively and adding, we get

$$(x + \lambda_1 a + \lambda_2 a')dx + (y + \lambda_1 b + \lambda_2 b')dy + (z + \lambda_1 c + \lambda_2 c')dz = 0.$$

Now, equating to zero, the coefficients of dx, dy and dz, we get

$$x + \lambda_1 a + \lambda_2 a' = 0 \quad ...(4)$$

$$y + \lambda_1 b + \lambda_2 b' = 0 \quad ...(5)$$

$$z + \lambda_1 c + \lambda_2 c' = 0 \quad ...(6)$$

Multiplying (4), (5) and (6) by x, y, z respectively and then adding, we get

$$(x^2 + y^2 + z^2) + \lambda_1(ax + by + cz) + \lambda_2(a'x + b'y + c'z) = 0$$

$$\Rightarrow \quad u + \lambda_1 . 1 + \lambda_2 . 1 = 0 \Rightarrow u + \lambda_1 + \lambda_2 = 0 \quad ...(7)$$

Again, multiplying (4), (5) and (6) by a, b, c respectively and adding, we get

$$(ax + by + cz) + \lambda_1(\Sigma a^2) + \lambda_2(\Sigma aa') = 0$$

$$\Rightarrow \quad 1 + \lambda_1(\Sigma a^2) + \lambda_2(\Sigma aa') = 0 \quad ...(8)$$

Again, multiplying (4), (5) and (6) by a', b' and c' respectively and adding, we get

$$(a'x + b'y + c'z) + \lambda_1(\Sigma aa') + \lambda_2(\Sigma a'^2) = 0$$

$$\Rightarrow \quad 1 + \lambda_1(\Sigma aa') + \lambda_2(\Sigma a'^2) = 0 \quad ...(9)$$

Now, eliminating λ_1 and λ_2 from (7), (8) and (9), we get

$$\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma a^2 & \Sigma aa' \\ 1 & \Sigma aa' & \Sigma a'^2 \end{vmatrix} = 0.$$

which gives the maxima or minima values of u.

EXAMPLE 4. *Find the extreme values of the function*

$$u = x^2 + y^2 + z^2 \qquad \ldots(1)$$

subject to two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad \ldots(2)$$

$$lx + my + nz = 0 \qquad \ldots(3)$$

SOLUTION : Geometrically the problem means the same as that of finding out the lengths of the axes of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$.

Differentiating (1), we get

$$du = 2xdx + 2ydy + 2zdz.$$

Since u is an extreme value, therefore $du = 0$.

$$\therefore \qquad xdx + ydy + zdz = 0 \qquad \ldots(4)$$

From (2) and (3),

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0$$

$$\Rightarrow \qquad \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \qquad \ldots(5)$$

and $$ldx + mdy + ndz = 0 \qquad \ldots(6)$$

Multiplying (4) by (1), (5) by λ_1 and (6) by λ_2, we get

$$\left(x + \frac{\lambda_1 x}{a^2} + \lambda_2 l\right)dx + \left(y + \lambda_1 \frac{y}{b^2} + \lambda_2 m\right)dy + \left(z + \lambda_1 \frac{z}{c^2} + \lambda_2 n\right)dz = 0.$$

Since dx, dy, dz are independent to each other, therefore we have

$$x + \lambda_1 \frac{x}{a^2} + \lambda_2 l = 0 \qquad \ldots(7)$$

$$y + \lambda_1 \frac{y}{b^2} + \lambda_2 m = 0 \qquad \ldots(8)$$

$$z + \lambda_1 \frac{z}{c^2} + \lambda_2 n = 0 \qquad \ldots(9)$$

We need to eliminate λ_1 and λ_2 from the equations (7), (8) and (9).

Multiplying the equations (7), (8) and (9) by x, y and z respectively and then adding, we get

$$(x^2 + y^2 + z^2) + \lambda_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \lambda_2 (lx + my + nz) = 0$$

$$\Rightarrow \quad u + \lambda_1 (1) + \lambda_2 (0) \Rightarrow x = -\lambda_1.$$

Hence, from (7), we have

$$x\left(1 - \frac{u}{a^2}\right) = -\lambda_2 l \Rightarrow x\left(\frac{u - a^2}{a^2}\right) = \lambda_2 l$$

$$\therefore \quad x = \frac{\lambda_2 a^2 l}{u - a^2}.$$

Similarly, from (8) and (9), we have

$$y = \frac{\lambda_2 b^2 m}{u - b^2} \quad \text{and} \quad z = \frac{\lambda_2 c^2 n}{u - c^2}.$$

But $lx + my + nz = 0$, therefore

$$\frac{\lambda_2 a^2 l^2}{u - a^2} + \frac{\lambda_2 b^2 m^2}{u - b^2} + \frac{\lambda_2 c^2 n^2}{u - c^2} = 0$$

$$\Rightarrow \quad \frac{a^2 l^2}{u - a^2} + \frac{b^2 m^2}{u - b^2} + \frac{c^2 n^2}{u - c^2} = 0; \text{ since } \lambda_2 \neq 0.$$

which is a quadratic in u, giving the extreme values of u.

EXAMPLE 5. *Prove that of all the rectangular parallelopipeds of the same volume the cubic has the least surface.* **(M.U. 67H)**

SOLUTION : Let the lengths of three coterminous edges of the rectangular parallelopiped be xyz. Then its volume V is given by $V = xyz$.

By question, $V = xyz = \text{const (say) } c$ ...(1)

The surface S is given by

$$S = 2(xy + yz + zx) \qquad \ldots(2)$$

We have to find the min.. value of (2) subject to the condition (1).

For max. or min. value of (2), we have

$$dS = 0 \Rightarrow (xdy + ydx) + (ydz + zdx) + (zdx + xdz) = 0$$

$$\Rightarrow (y + z)dx + (z + x)dy + (x + y)dz = 0 \qquad \ldots(3)$$

Again, differentiating (1),

$$yzdx + zxdy + xydz = 0. \qquad ...(4)$$

Now, multiplying (3) by (1) and (4) by λ, and adding them, we get

$$(y + z + \lambda yz)dx + (z + x + \lambda zx)dy + (x + y + \lambda xy)dz = 0.$$

Equating to zero, the coefficients of dx, dy and dz, we get

$$y + z + \lambda yz = 0 \qquad ...(5)$$

$$z + x + \lambda zx = 0 \qquad ...(6)$$

$$x + y + \lambda xy = 0 \qquad ...(7)$$

From (5), (6) and (7), we have

$$-\lambda = \frac{y+z}{yz} = \frac{z+x}{zx} + \frac{x+y}{xy}$$

$$\Rightarrow \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{y} = \frac{1}{z}; \text{ taking two by two}$$

$$\Rightarrow \quad x = y = z.$$

Thus, for max. or min. value of S, $x = y = z$.

Now, we have to find out whether S is max. or min.

Differentiating (1) viz. $xyz = c$ partially, regarding z as a function of x and y, we get

$$y\left\{x\frac{\partial z}{\partial x} + z\right\} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}.$$

Similarly $\dfrac{\partial z}{\partial y} = -\dfrac{z}{y}$.

Now, from (2) $\dfrac{\partial S}{\partial x} = 2\left\{y + y\dfrac{\partial z}{\partial x} + \left(z.1 + x\dfrac{\partial z}{\partial x}\right)\right\}$

$$= 2\left\{y + z + (y + x)\left(-\frac{z}{x}\right)\right\}$$

$$= 2\left\{y + z - \frac{yz}{x} - z\right\} = 2y - \frac{2yz}{x};$$

$$\frac{\partial S}{\partial y} = 2\left\{x + \left(y\frac{\partial z}{\partial y} + z\right) + x\frac{\partial z}{\partial y}\right\}$$

$$= 2\left\{x + z + y\left(-\frac{z}{y}\right) + x\left(-\frac{z}{y}\right)\right\}$$

$$= 2\left\{x + z - z - \frac{xz}{y}\right\} = 2x - \frac{2xz}{y}.$$

$$r = \frac{\partial^2 S}{\partial x^2} = 0 - 2y\left\{\frac{1}{x}\cdot\frac{\partial z}{\partial x} + z\left(-\frac{1}{x^2}\right)\right\}$$

$$= -2y\left\{\frac{1}{x}\left(-\frac{z}{x}\right) - \frac{z}{x^2}\right\} = -2y\left(\frac{-2z}{x^2}\right)$$

$$= \frac{4yz}{x^2} = 4 \text{ when } x = y = z.$$

$$s = \frac{\partial^2 S}{\partial x \partial y} = 2 - \frac{2}{y}\left\{x\frac{\partial z}{\partial x} + z\right\} = 2 - \frac{2}{y}\left\{x\left(-\frac{z}{x}\right) + z\right\}$$

$$= 2 - \frac{2}{y}(-z + z) = 2$$

$$t = \frac{\partial^2 S}{\partial y^2} = 0 - 2x\left\{\frac{1}{y}\cdot\frac{\partial z}{\partial y} + z\left(-\frac{1}{y^2}\right)\right\}$$

$$= -2x\left\{\frac{1}{y}\left(-\frac{z}{y}\right) - \frac{z}{y^2}\right\} = -2x\left(\frac{-2z}{y^2}\right)$$

$$= \frac{4xz}{y^2} = 4 \text{ when } x = y = z.$$

Therefore $rt - s^2 = 16 - 4 = 12 > 0$ and $r > 0$. Hence the surface is least when $x = y = z$ i.e. when the parallelopiped is a cubic.

EXAMPLE 6. *Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \textbf{(P.U. 62H; R.U. 67H, 69H)}$$

SOLUTION : Let (x, y, z) be the co-ordinates of any vertex of the rectangular parallelopied inscribed in the given ellipsoid. Then the lengths of the coterminous edges of the rectangular parallelopiped will be $2x$, $2y$, $2z$, so that its volume V is $= 8xyz$.

Thus we have to find the max. value of

$$V = 8xyz \qquad \ldots(1)$$

subject to $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \ldots(2)$$

Differentiating (1), we have

$$\frac{1}{8}dV = yzdx + zxdy + xydz$$

For max. or min. $dV = 0$.

$$\therefore \quad yzdx + zxdy + xydz = 0$$

$$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \qquad ...(3)$$

Again, differentiating (2), we get

$$\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \qquad ...(4)$$

Multiplying (3) by (1) and (4) by λ and adding them, we get

$$\left(\frac{1}{x} + \lambda \cdot \frac{x}{a^2}\right)dx + \left(\frac{1}{y} + \lambda \cdot \frac{y}{b^2}\right)dy + \left(\frac{1}{z} + \lambda \cdot \frac{z}{c^2}\right)dz = 0.$$

Equating to zero, the coefficients of dx, dy and dz, we get

$$\frac{1}{x} + \lambda \cdot \frac{x}{a^2} = 0 \qquad ...(5)$$

$$\frac{1}{y} + \lambda \cdot \frac{y}{b^2} = 0 \qquad ...(6)$$

$$\frac{1}{z} + \lambda \cdot \frac{z}{c^2} = 0 \qquad ...(7)$$

Multiplying (5), (6) and (7) by x, y, z respectively and then adding, we get

$$3 + \lambda.1 = 0 \Rightarrow \lambda = -3.$$

$\therefore$ From (5), $\frac{1}{x} - 3\frac{x}{a^2} = 0 \Rightarrow 3x^2 = a^2 \therefore x = \frac{a}{\sqrt{3}}$.

Similarly $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$.

$$\therefore \quad V = 8xyz = 8 \times \frac{a}{\sqrt{3}} \times \frac{b}{\sqrt{3}} \times \frac{c}{\sqrt{3}} = \frac{8}{3\sqrt{3}}abc.$$

Now, we have to find out whether this volume is max or min.

Differentiating (2), partially with respect to x, treating z as a function of x, y we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2 z}.$$

Now $$\frac{\partial V}{\partial x} = 8y\left\{x\cdot\frac{\partial z}{\partial x} + z\right\} = 8y\left\{x\left(-\frac{c^2x}{a^2z}\right) + z\right\}$$

$$= 8y\left\{z - \frac{c^2x^2}{a^2z}\right\}$$

$$\Rightarrow \quad r = \frac{\partial^2 V}{\partial x^2} = 8y\left[\frac{\partial z}{\partial x} - \frac{c^2}{a^2}\left\{\frac{1}{z}\cdot 2x + x^2\left(-\frac{1}{x^2}\right)\frac{\partial z}{\partial x}\right\}\right]$$

$$= 8y\left[-\frac{c^2x}{a^2z} - \frac{c^2}{a^2}\left\{\frac{2x}{z} - \frac{x^2}{z^2}\cdot\left(-\frac{c^2x}{a^2z}\right)\right\}\right]$$

$$= 8y\left[-\frac{c^2x}{a^2z} - \frac{2c^2x}{a^2z} - \frac{c^4}{a^4}\frac{x^3}{z^3}\right]$$

$$= 8y\left[-\frac{3c^2x}{a^2z} - \frac{c^4}{a^4}\frac{x^3}{z^3}\right]$$

$$= -8y\left[\frac{3c^2x}{a^2z} + \frac{c^4}{a^4}\frac{x^3}{z^3}\right]$$

which is clearly negative for $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$.

Hence V is max. and its value is $\frac{8}{3\sqrt{3}}abc$.

Example 38

1. Find the minimum value of $u = x^2 + y^2 + z^2$ where $xyz = a^2$.

2. Prove that the maximum and minimum values of $x^2 + y^2$ where $ax^2 + 2hxy + by^2 = 1$ are given by the roots of the quadratic.

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2. \qquad \textbf{(Bhag. 67H, 69H; M.U. 71H)}$$

3. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions $lx + my + nz = 0$ and $ax^2 + by^2 + cz^2 = 1$. **(B.U. 53H; M.U. 70H)**

4. Find the maximum or minimum values of $u = a^2x^2 + b^2y^2 + c^2z^2$ when $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$. **(Bhag. 69H, R.U. 71H)**

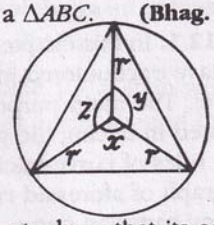
5. Find the max. and min. values of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ subject to the conditions $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **(P.U. 53H; B.U. 57H; M.U. 83H)**

6. If $(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$ and $lx + my + nz = 0$, show that the maximum and minimum values of $r^2 = x^2 + y^2 + z^2$ are given by the equation

$$\frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0. \qquad \textbf{(P.U. 59H; R.U. 66H)}$$

7. Find the maximum value of $\sin A \sin B \sin C$ for a ΔABC. **(Bhag. 72H)**

8. Prove that the maximum triangle which can be inscribed in a circle is equilateral.



[Hint : $x + y + z = 2\pi$

Area $= \frac{1}{2}r^2(\sin x + \sin y + \sin z)$.**]**

9. Given that the perimeter of a triangle is constant, show that its area is minimum when it is equilateral.

[Hint : If u be the area, then

$$u^2 = s(s - a)(s - b)(s - c), \text{ where } 2s = a + b + c.\text{]}$$

10. Prove that the rectangular solid of maximum volume for given surface area is a cube.

11. Prove that the rectangular solid of maximum volume which can be inscribed in a given sphere is a cube.

12. If $xyz = 8$, find the minimum value of the product

$$(x + 1)(y + 1)(z + 1).$$

13. Prove that if $x + y + z = 1$, $ayz + bzx + cxy$ has a extreme value equal to

$$\frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

ANSWERS

1. $3a^2$ **3.** $\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0$ where $u = x^2 + y^2 + z^2$.

4. $\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0.$

5. $\frac{l^2a^4}{a^2u - 1} + \frac{m^2b^4}{b^2u - 1} + \frac{n^2c^4}{c^2u - 1} = 0$ where $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$

7. $\frac{3\sqrt{3}}{8}$ **12.** 9.

12

CURVE-TRACING

12.1. In this chapter we shall deal with the graph of those curves which we have encountered in the chapter on ' Tangents and Normals ' or in 'Curvature'. The main purpose of this chapter is to point out those rules which are used in tracing the graph of a curve. At the outset, we shall describe the main rules of curve-tracing and then afterwards we shall use them in tracing the graph of aforesaid curves. First of all, we shall describe the rules of tracing any cartesian curve.

12.2. Sec. I. Rules for tracing of a cartesian curve.

(i) The curve passes through the origin :

Firstly we should see whether the curve passes through the origin or not. For this, we put $x = 0$, $y = 0$ in the equation of the given curve. If both the sides of the equation of the curve are satisfied, then we say that the curve passes through the origin (0, 0).

For example, the curve $y^2 = 4ax$ (parabola) passes through the origin (0, 0).

(ii) Symmetry :

After this, we consider the symmetry of a given curve. The symmetry of a curve may occur in the following important ways.

(a) If the equation of the curve contains only even powers of y (so that the equation of the curve does not change by putting $-y$ for y in it), then the curve is said to be symmetrical about the x-axis.
When we say the curve is symmetrical about the x-axis, it means that the portion of the curve above the x-axis will be exactly the same below the x-axis.

(b) If the equation of the curve contains only even powers of x, then the curve is said to be symmetrical about the y-axis.

(c) If the equation of the curve contains only even powers of x and y, then the curve is said to be symmetrical about both the axes; e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetrical about both the axes.

(d) If the equation of the curve does not change by putting y for x and x for y in the given equation, then we say that the curve is symmetrical about the line $y = x$. For example, $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

(iii) Point of intersection of the curve with the co-ordinate axes:

Now we should see at what points the curve cuts the x-axis and y-axis. In order to determine the co-ordinates of the point of intersection with the x-axis we shall put $y = 0$ in the equation of the curve because the y-co-ordinate of any point situtated on the x-axis is = 0.

Similarly, in order to find out the co-ordinates of the point of intersection with the y-axis we shall put $x = 0$ in the equation of the curve.

Note : If the curve is symmetrical about the line $y = x$, then we should find out the co-ordinates of the point of intersection of the curve with $y = x$.

(iv) Region or regions of the curve :

We shall solve for y from the equation of the given curve. If, for example, suppose that by giving to x, the values $x > a$ or $x < -a$, the values of y become imaginary, then it means that there would be no portion of the curve on the R.H.S. of $x = a$ or there would be no portion of the curve on the L.H.S. of $x = -a$.

Note : Sometimes it happens that the value of y increases corresponding to increasing values of x. In this case, the extent of the given curve will be up to infinity.

(v) Equation of tangent at the origin :

If the curve passes through the origin, then we should find out the equation of the tangent to the curve at the origin. In order to find out the equation of the tangent at the origin, the formula is to equate to zero the terms of the lowest degree from the equation of the curve. For example, in the curve $y^2 = 4ax$, the term of the lowest degree is $4ax$. Therefore $x = 0$ i.e. y-axis is the equation of the tangent at the origin to the curve (parabola) $y^2 = 4ax$.

Note : This will also enable us to locate the 'double point' if any, of the given curve.

(vi) Asymptotes, *if any* :

For those curves, which possess asymptotes, we should also find out their asymptotes by applying the appropriate rules.

To sum up, the aforesaid rules in general are observed for drawing the graph of a cartesian curve.

12.3. Sec II. Rules for tracing of a polar curve.

(i) The curve passes through the pole :

If $r = 0$ for any real value of θ; we say that the curve passes through the origin.

(ii) Symmetry :

(a) If by putting—θ for θ, the equation of the curve does not change, we say that the curve is symmetrical about the initial line OX.

(b) If by putting π—θ for θ, the equation of the curve does not change, we say that the curve is symmetrical about the line $\theta = \pi/2$ i.e. about the line OY.

(c) If by putting—r for r, the equation of the curve does not change; i.e. if the equation contains only the even powers of r, we say that the curve is symmetrical about the pole.

(iii) Intersection of the curve with the initial line OX and also with OY.

For this, we find out the value of r by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in the equation of the curve.

(iv) Region of the curve :

We also determine those regions of the curve which do not contain any portion of the curve and they are determined by those values of θ for which r is imaginary.

(v) Tangent at the pole :

If the curve passes through the origin, then we get the equation of the tangent at the origin corresponding to those values of θ for which $r = 0$.

(vi) Asymptotes, *if any* :

If the curve extends up to infinity, then we should also find out the asymptote of the curve from the formula in the polar form. Otherwise, it is convenient to determine the asymptote by changing the equation of the curve in cartesian form.

(vii) Some more points :

(a) We should plot some special points on the curve. For this, we take some special values of θ, say $\theta = 0$, $\theta = \pi/6$, $\theta = \pi/4$, $\theta = \pi/3$, $\theta = \pi/2$ .... and after determining the corresponding values of r we plot these points on the curve.

(b) Sometimes it is convenient to trace out a curve by transforming the curve in the cartesian form.

(c) As per our requirement, we find out the angle ϕ between the radius vector and the tangent by using the formula $\tan\phi = rd\theta/dr$ so that we can determine the co-ordinates of those points where $\phi = \pi/2$ is any specified angle.

Now we use the aforesaid rules of curve-tracing in drawing the graph of some special curves.

12.4. Some well-known curves

I. Curve $x^{2/3} + y^{2/3} = a^{2/3}$ (Astroid)

(i) Since by putting $x = 0$, $y = 0$, both the sides of the curve are not satisfied, hence the curve does not pass through the origin.

(ii) The equation of the given curve does not change when we put—x for x and—y for y in it and therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get $x^{2/3} = a^{2/3}$ or $x^2 = a^2$ $\therefore$ $x = \pm a$ i.e. the curve cuts the x-axis at the points $(a, 0)$ and $(-a, 0)$. Similarly putting $x = 0$, we get $y^{2/3} = a^{2/3}$ $\therefore$ $y = \pm a$ i.e. the curve cuts the y-axis at the points $(0, a)$ and $(0, -a)$.

(iv) Again from the equation of the curve $y^{2/3} = a^{2/3} - x^{2/3}$.

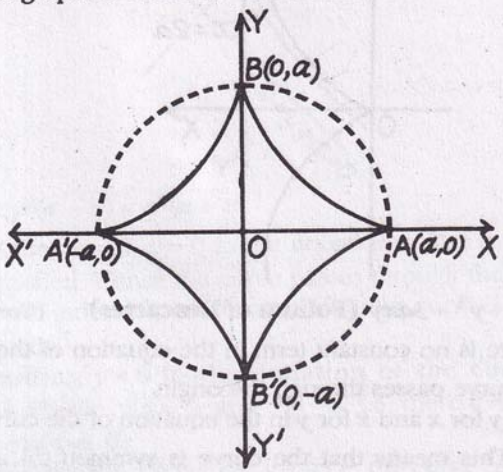
$\therefore$ If we put in this equation any quantity $x > a$ or $x < -a$, then $y^{2/3} = -ve$ $\therefore$ $y^2 = -ve$.

Therefore y becomes imaginary i.e., there is no portion of the curve on the R.H.S. of $x = +a$ or on the L.H.S. of $x = -a$. Similarly by solving for x from the given equation, it can be shown that there is no portion of the curve above $y = +a$ or below $y = -a$.

That is, the given curve is enclosed by four points $(a, 0)$, $(-a, 0)$, $(0, a)$ and $(0, -a)$.

Also, by comparing the curve $x^{2/3} + y^{2/3} = a^{2/3}$ with the circle $x^2 + y^2 = a^2$ we find that since $\frac{2}{3} < 2$, the given curve will be inside the circle $x^2 + y^2 = a^2$.

Hence the graph of the curve will be like this :



II. Curve $y^2(2a - x) = x^3$

(i) By putting $x = 0$, $y = 0$ in the equation of the curve, both the sides of the curve are satisfied, therefore the curve passes through the origin.

(ii) Since the equation of the curve contains only even powers of y, therefore the curve is symmetrical about x-axis.

(iii) By putting $y = 0$ in the equation of the curve, we find that $x = 0$. Similarly by putting $x = 0$ we find that $y = 0$. This means that the curve does not cut the co-ordinate axes at any other point except at the origin.

(iv) By equating to zero the term of the lowest degree from the equation of the curve, viz $2ay^2 = 0$, we find that $y = 0$. That is, x-axis ($y = 0$), is the tangent to the curve at the origin.

(v) By equating to zero the coefficient of the term of the highest degree in y we find $(2a - x) = 0$ i.e. $x = 2a$. Therefore the asymptote of the curve parallel to y-axis is $x = 2a$.

(vi) Again, solving for y from the equation of the curve, we get

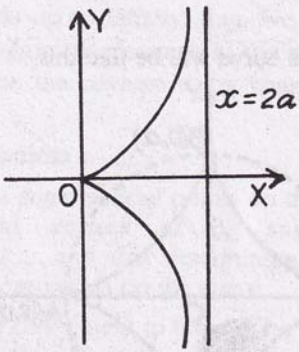
$$y^2 = \frac{x^3}{2a - x}.$$

Now if x is $-ve$, then $y^2 = -ve$ $\therefore$ y is imaginary

and if $x > 2a$, then $y^2 = -ve$ $\therefore$ y is imaginary.

Therefore there will be no portion of the curve on the L.H.S. of $x = 0$ or on the R.H.S. of $x = 2a$.

Hence the graph of the curve will be as follows :



III. Curve $x^3 + y^3 = 3axy$ (Folium of Descartes) **(Gorakhpur 87, 89)**

(i) Since there is no constant term in the equation of the curve, therefore the curve passes through the origin.

(ii) If we put y for x and x for y in the equation of the curve, it does not change. This means that the curve is symmetrical about the line $y = x$.

(iii) By equating to zero the term of the lowest degree in the equation of the curve we get $3axy = 0$ which $\Rightarrow x = 0, y = 0$ i.e. the equations of the tangents at the origin are $x = 0$ and $y = 0$.
Thus origin is a double point.

(iv) By putting $y = 0$ in the equation of the curve we get $x = 0$ and similarly by putting $x = 0$ we get $y = 0$. Hence the curve cuts both the co-ordinates axes only at the origin.

(v) x and y cannot be negative simultaneously, for in that case L.H.S. $= -ve$ and R.H.S. $= +ve$.
Thus there will be no portion of the curve in the third quadrant (where $x = -ve$ and $y = -ve$).

(vi) The only real asymptote of the curve is $x + y + a = 0$. (this has been derived in the chapter on asymptote in page 331).

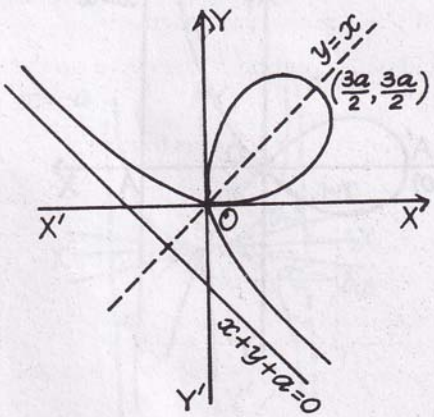
(vii) This curve cuts the line $y = x$ in those points where $x^3 + y^3 = 3axy$.
This $\Rightarrow x^3 + x^3 = 3ax.x \Rightarrow 2x^3 = 3ax^2$

$$\Rightarrow x^2(2x - 3a) = 0, \quad \therefore x = 0, x = \frac{3a}{2}.$$

That is, the curve cuts the line $y = x$ in two points

$$(0, 0) \text{ and } \left(\frac{3a}{2}, \frac{3a}{2}\right).$$

Hence the graph of the curve is as follows :



IV. Curve $y^2(a - x) = x^2(a + x)$.

(i) By putting $x = 0, y = 0$ in the equation of the curve both the sides are satisfied. Hence the curve passes through the origin.

(ii) Since the index of y is even, therefore the curve is symmetrical about the x-axis.

(iii) By putting $y = 0$ in the equation of the curve we find that $x^2(a + x) = 0 \quad \therefore x = 0$, and $x = -a$, i.e. the curve cuts the y-axis at the point $(-a, 0)$.
Again putting $x = 0$ we get $y = 0$ i.e. the curve cuts the y-axis at the point $(0, 0)$.

(iv) From the equation of the given curve, we get

$$y^2 = \frac{x^2(a+x)}{a-x}$$

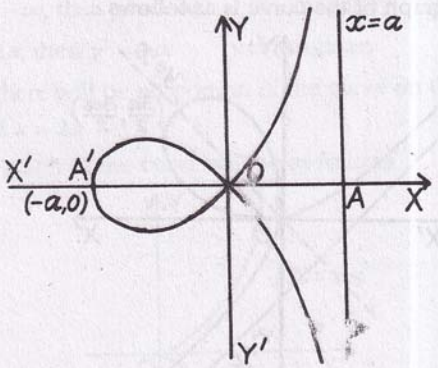
$$\Rightarrow \qquad y = x\sqrt{\frac{a+x}{a-x}}.$$

Now if in this equation we give to x values such that $x < -a$ or $x > a$ then y becomes imaginary. Therefore there will be no portion of the curve on the L.H.S. of $x = -a$ or on the R.H.S. of $x = a$.

(v) By equating to zero the term of the lowest degree in the equation of the curve we get $y^2 - x^2 = 0$ i.e. $y = \pm x$.
That is, the equation of the tangent at the origin is $y = \pm x$.

(vi) By equating to zero the coefficient of the term containing the highest power of y we get $a - x = 0$.
Hence the eqation of the asymptote parallel to y-axis is $x = a$.

(vii) At the point $(-a, 0)$ we have $\frac{dy}{dx} = \infty$, i.e. at the point $(-a, 0)$. the tangent will be perpendicular to x-axis. Hence the graph of the curve will be as follows :



V. Curve $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

The given curve can be written in the following form :

$$a^2y^2 - b^2x^2 = x^2y^2.$$

(i) Since putting $x = 0$, $y = 0$, both the sides of the equation are satisfied, therefore the curve passes through the origin.

(ii) Since the indices of x and y both are even, therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get $x = 0$. Similarly putting $x = 0$ we get $y = 0$. This means that the curve does not cut the axes elsewhere except at the origin.

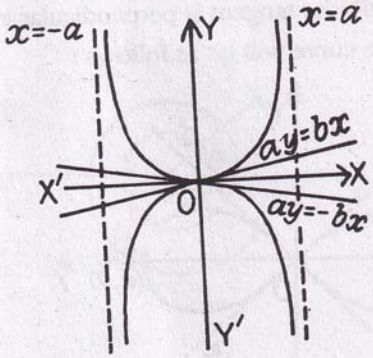
(iv) By equating to zero the term of the lowest degree in the equation of the curve we get $a^2y^2 - b^2x^2 = 0$.
This $\Rightarrow$ $ay = \pm bx$. That is, the equations of the tangent at the origin are $ay = \pm bx$.

(v) By equating to zero the coefficient of the term containing the highest degree in y in the equation of the curve, we have $x^2 - a^2 = 0$.
This $\Rightarrow$ $x = \pm a$.
Therefore the equations of the asymptote parallel to the y-axis are $x = \pm a$.
Similarly by equating to zero, the term containing the highest degree in y, we have $b^2 + y^2 = 0$ from which y is imaginary. That is, the asymptotes parallel to the x—axis are imaginary.

(vi) Solving for y from the equation of the curve we get

$$y^2(a^2 - x^2) = b^2x^2 \quad \therefore \quad y = \frac{bx}{\sqrt{a^2 - x^2}}.$$

Now, if we put $x > a$ or $x < -a$ in this equation, then y becomes imaginary. Therefore there is no portion of the curve outside $x = a$ or $x = -a$.
Hence the graph of the curve will be as follows :



VI. Curve $x^4 + y^4 = a^2(x^2 - y^2)$

(i) By putting $x = 0$, $y = 0$, in the equation of the curve we find that both the sides are satisfied. Hence the curve passes through the origin.

(ii) Since the equation of the curve contains only the even powers of x and y, therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get

$$x^4 - a^2x^2 = 0 \Rightarrow x^2(x^2 - a^2) = 0 \Rightarrow x = 0, \pm a.$$

That is, the curve cuts the x-axis at three points $(0, 0)$, $(a, 0)$, $(0, -a)$.

Again putting $x = 0$, we get $y^4 + a^2y^2 = 0$.

$\therefore$ $y = 0$ i.e. the curve cuts the y-axis at the origin only.

(iv) By equating to zero the term of the lowest degree in the equation of the curve we get $x^2 - y^2 = 0$ i.e. $y = \pm x$. Therefore the equations of the tangents at the origin will be $y = x$, $y = -x$.

(v) There is no asymptote of the curve.

(vi) Solving for y from the equation of the curve, we get

$$y^4 + a^2y^2 = a^2x^2 - x^4 = x^2(a^2 - x^2).$$

If we give to x values $x > a$ or $x < -a$, then $a^2 - x^2 = -ve$ (say—k), so that $y^2 + a^2y^2 + k = 0$. Since there is no change of sign according to Descarte rule, therefore the roots of y are neither positive nor negative and hence y is imaginary. That is, there is no portion of the curve on the R.H.S. of $x = a$ or on the L.H.S. of $x = -a$.

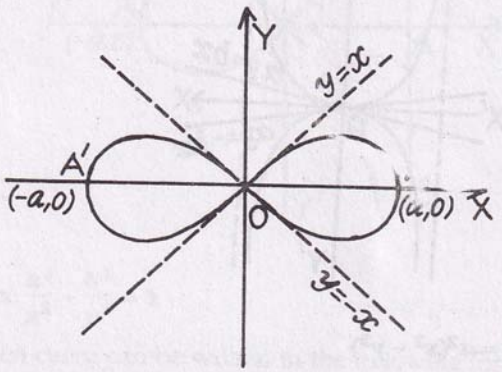
(vii) Differentiating the equation of the curve w.r.t. x, we get

$$4x^3 + 4y^3\frac{dy}{dx} = a^2\left(2x - 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}(4y^3 + 2a^2y) = 2a^2x - 4x^3 \quad \therefore \quad \frac{dy}{dx} = \frac{2a^2x - 4x^3}{4y^3 + 2a^2y}$$

$\therefore$ at the point $(a, 0)$, $\frac{dy}{dx} = \infty$ and also at $(-a, 0)$ we have $\frac{dy}{dx} = \infty$ i.e. at the points $(a, 0)$ and $(-a, 0)$, the tangent is perpendicular to the x-axis.

Hence the graph of the curve will be as follows :



VII. Curve $r = a\,e^{m\theta}$ (*Equiangular Spiral*)

(i) r is always positive and increases with θ and when $\theta \to \infty$ then $r \to \infty$.

(ii) r is not zero for any finite value of θ and hence pole is not situated on the curve.

(iii) If $\theta = 0$, then $r = a$;

" $\theta = \frac{\pi}{2}$, then $r = a\, e^{\frac{m\pi}{2}}$

" $\theta = \pi$, then $r = a\, e^{m\pi}$ ...

Thus by giving to θ values greater than 0, r becomes increasingly greater than a.

(iv) From the equation $r = ae^{m\theta}$ of the curve, we have

$$\frac{dr}{d\theta} = ae^{m\theta}.m = mr$$

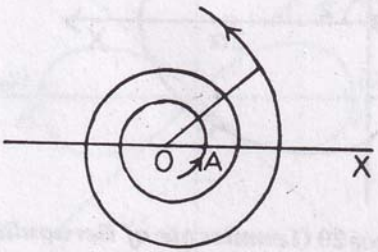
$$\therefore \quad \tan\phi = \frac{rd\theta}{dr} = \frac{r}{mr} = \frac{1}{m}$$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{1}{m}\right) = \text{constant}.$$

Hence at any point the angle between the radius vector and tangent is constant.

This is the reason why this curve is called *equiangular spiral* and its shape is just like a spiral.

Hence the graph of the curve will be like this :



VIII. Cardiode $r = a(1+\cos\theta)$

(i) If we put $-\theta$ for θ in the equation of the curve, we find that

$$r = a\{1 + \cos(-\theta)\} = a(1 + \cos\theta)$$

i.e. the equation of the curve does not change. Therefore the given curve is symmetrical about the initial line.

(ii) $r = 0$ when $1 + \cos\theta = 0$ i.e. $\cos\theta = -1$ $\therefore$ $\theta = \pi$. Hence the curve passes through the origin and the equation of the tangent at the pole is $\theta = \pi$ i.e. the inital line.

(iii) Now we plot some of the points on the curve.

when $\theta = 0$, then $r = 2a$;

when $\theta = \frac{\pi}{3}$, then $r = \frac{3}{2}a$;

when $\theta = \frac{\pi}{2}$, then $r = a$;

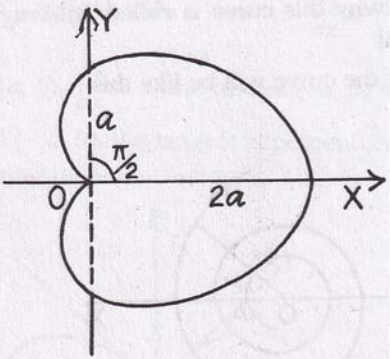
when $\theta = \pi$, then $r = 0$.

(iv) From the given equation $r = a(1 + \cos\theta)$, we have $\frac{dr}{d\theta} = -a\sin\theta$.

That is, when the value of θ increases from 0 to π, then the value of r decreases and, as has been stated in (iii), decreases from $2a$ to 0.

Again, since the given curve is symmetrical about the initial line, therefore when the value of θ increases from π to 2π, then the value of r increases from 0 to $2a$.

Hence the graph of the curve will be like this :



IX. Curve $r^2 = a^2\cos 2\theta$ (*Lemniscate of Bernaulli*)

(i) (a) If we put—θ for θ in the equation of the curve, then the equation of the curve does not change. Therefore the curve is symmetrical about the initial line.

(b) If we put—r for r in the equation of the curve, the equation of the curve does not change. Therefore the curve is symmetrical about the pole.

(c) If we put π—θ for θ in the equation of the curve, then

$$r^2 = a^2\cos 2(\pi - \theta) = a^2\cos 2\theta$$

i.e. the equation of the curve does not change. Therefore the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(ii) $r = 0$ when $\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

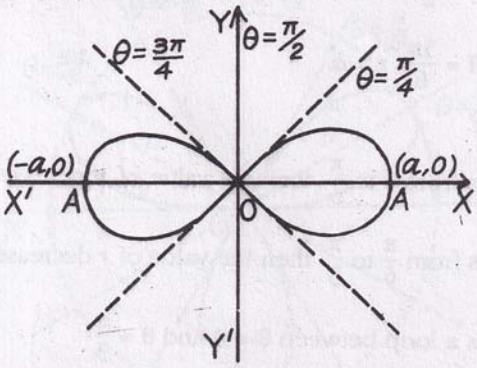
Hence the curve passes through the pole and the equations of the tangents at the pole are $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

(iii) $\theta = 0 \Rightarrow r^2 = a^2 \quad \therefore \quad r = \pm a$. Hence the points $(a, 0)$, $(-a, 0)$ are situated on the curve.

(iv) When the value of θ increases from 0 to $\frac{\pi}{4}$, the value of r^2 decreases from a^2 to 0. Again, when the value of θ increases from $\frac{3\pi}{4}$ to π, the value of r^2 increases from 0 to a^2. But when the value of θ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, then $r^2 = -ve$.

$\therefore \quad r$ becomes imaginary. That, is no portion of the curve lies between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Hence the graph of the curve will be like this—



Note : Changing the equation of the curve in cartestian form, we get

$$r^2 = a^2(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \quad r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$

But $\quad x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore \quad x^2 + y^2 = r^2.$$

Hence $\quad (x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Then its graph can be drawn like a cartesian curve.

X. Curve $r = a \sin 3\theta$

(i) If we put π—θ for θ in the equation of the curve we find that

$$r = a\sin3(\pi - \theta)$$
$$= a\sin(3\pi - 3\theta) = a\sin(2\pi + \pi - 3\theta)$$
$$= a\sin(\pi - 3\theta) = a\sin3\theta.$$

That is, the equation of the curve does not change. Therefore the given curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(ii) $r = 0$ when $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi \therefore \theta = \frac{n\pi}{3}$

i.e. $r = 0$ when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \ldots$

Hence the curve passes through the pole and the equations of the tangents at the pole are

$$\theta = 0, \ \theta = \frac{\pi}{3}, \ \theta = \frac{2\pi}{3};$$

and when $\theta = \frac{\pi}{6}$, $r = a$;

when $\theta = \frac{\pi}{2}$, $r = -a$;

when $\theta = \frac{5\pi}{6}$, $r = a$.

(iii) We see that

(a) If θ increases from 0 to $\frac{\pi}{6}$, then the value of r increases from 0 to a;

if θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, then the value of r decreases from a to 0.

Thus there is a loop between $\theta = 0$ and $\theta = \frac{\pi}{3}$.

(b) If θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, then the value of r becomes $-a$ from 0;

if θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, then the value of r becomes 0 from $-a$.

Thus there is another loop between $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

(c) If θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, then the value of r becomes a from 0;

if θ increases from $\frac{5\pi}{6}$ to π, then the value of r becomes 0 from a.

Thus there is a third loop between $\theta = \frac{2\pi}{3}$ and $\theta = \pi$.

(iv) From the given curve, $\frac{dr}{d\theta} = 3a\cos 3\theta$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{a\sin 3\theta}{3a\cos 3\theta} = \frac{1}{3}\tan 3\theta$$

$$\therefore \quad \phi = 0 \Rightarrow \tan 3\theta = 0 \Rightarrow 3\theta = n\pi.$$

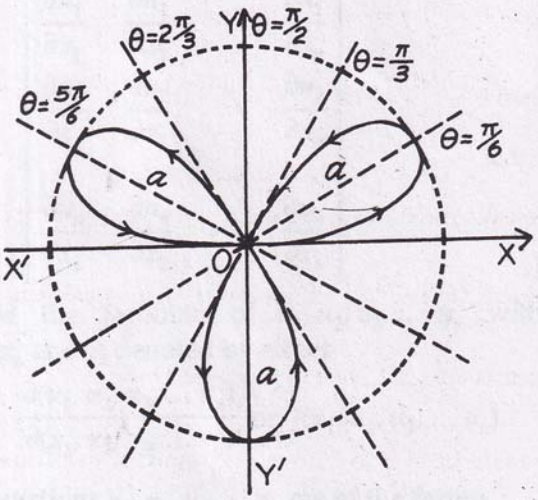
i.e. $\quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi.$

and $\quad \phi = \frac{\pi}{2} \Rightarrow \cot 3\theta = 0 \Rightarrow \cos 3\theta = 0$

$$\Rightarrow \quad 3\theta = (2n+1)\frac{\pi}{2} \quad \therefore \quad \theta = (2n+1)\frac{\pi}{6}$$

i.e. $\quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}.$

Hence the graph of the curve will be like this—



Example 39

Trace the following curves—

1. $a^2y^2 = x(x-a)^2$
2. $3ay^2 = x^2(a-x)$
3. $27ay^2 = 4(x-2a)^3$
4. $x^3 + y^3 = 2ax^2$ **(P.U. 57A)**
5. $y^2(a+x) = x^2(a-x)$
6. $y^2(a+x) = x^2(3a-x)$

7. $x(x^2 + y^2) = a(x^2 - y^2)$

8. $a^2y^2 = x^2(a^2 - x^2)$

9. $y^2 = \dfrac{x^2(1 + x^2)}{1 - x^2}$

10. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

11. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$

12. $x^4 + y^4 = 4a^2xy$ **(Gorakhpur 88)**

13. $x^2 = y^2(x + 1)^3$ **(Gorakhpur 92)**

14. $y^2 = (x - 1)(x - 2)(x - 3)$

15. $r = a \sin 2\theta$

16. $r^2 = a^2 \cos 2\theta$ **(P.U. 57S)**

17. $r^3 = a^3 \cos 3\theta$ **(P.U. 58S)**

18. $r = a + b \cos\theta$

CHAPTER

13

Jacobians

13.1 Definition

If $u_1, u_2, u_3, \ldots u_n$ are differentiable functions of n variables $x_1, x_2, x_3, \ldots x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, u_3, \ldots, u_n$ with respect to $x_1, x_2, x_3, \ldots x_n$ and is denoted by either

$$\frac{\partial(u_1, u_2, u_3, \ldots u_n)}{\partial(x_1, x_2, x_3, \ldots x_n)} \text{ or } J(u_1, u_2, u_3, \ldots u_n).$$

13.2 If the functions $u_1, u_2, u_3, \ldots, u_n$ are of the forms

$$u_1 = f_1(x_1),\ u_2 = f_2(x_1, x_2),\ u_3 = f_3(x_1, x_2, x_3), \ldots$$

$$u_n = f_n(x_1, x_2, x_3, \ldots x_n), \text{ then } \frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots \frac{\partial u_n}{\partial x_n}.$$

Soln. We have $\dfrac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

In such cases, the Jacobian reduces to the product of its leading terms.

Ex.1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$.

Soln. We have, $\dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta) + r\sin\theta\,(\sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\,(\cos^2\theta + \sin^2\theta) = r \cdot 1 = r.$$

Ex.2. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

Soln. We have, $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2 \sin\theta \cos\theta \cos^2\phi + r^2 \sin\theta \cos\theta \sin^2\phi)$$
$$+ r \sin\theta\,(r \sin^2\theta \cos^2\phi + r \sin^2\theta \sin^2\phi);$$
expanding along the third row
$$= \cos\theta \cdot r^2 \sin\theta \cos\theta\,(\cos^2\phi + \sin^2\phi)$$
$$+ r \sin\theta\, r \sin^2\theta\,(\cos^2\phi + \sin^2\phi)$$
$$= r^2 \sin\theta \cos^2\theta + r^2 \sin^3\theta$$
$$= r^2 \sin\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= r^2 \sin\theta \cdot 1 = r^2 \sin\theta.$$

Ex.3. If $u_1 = \dfrac{x_2x_3}{x_1}$, $u_2 = \dfrac{x_3x_1}{x_2}$, $u_3 = \dfrac{x_1x_2}{x_3}$, **prove that** $J(u_1, u_2, u_3) = 4$.

Soln. We have, $J(u_1, u_2, u_3)$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{x_2x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3x_1}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{(x_2x_3)(x_3x_1)(x_1x_2)}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix};$$

applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + R_1$

$$= -1 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -1(0-4) = 4.$$

Ex.4. If $x_1 = \cos u_1, x_2 = \cos u_2 \sin u_1, x_3 = \cos u_3 \sin u_2 \sin u_1$, show that

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = -\sin^3 u_1 \sin^2 u_2 \sin u_3.$$

Soln. Following the Art 13.2, we have

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = \frac{\partial x_1}{\partial u_1} \cdot \frac{\partial x_2}{\partial u_2} \cdot \frac{\partial x_3}{\partial u_3}$$

$$= (-\sin u_1)\{\sin u_1 (-\sin u_2)\}\{\sin u_2 \cdot \sin u_1 (-\sin u_3)\}$$

$$= -\sin^3 u_1 \sin^2 u_2 \sin u_3.$$

Ex.5. Find the Jacobian of $y_1, y_2, y_3, \ldots y_n$, being given

$$y_1 = 1 - x_1, y_2 = x_1(1 - x_2), y_3 = x_1 x_2 (1 - x_3), \ldots$$

$$y_n = x_1 x_2 x_3 \ldots x_{n-1}(1 - x_n).$$

Soln. Here functions $y_1, y_2, y_3, \ldots y_n$ are of the form as given in the art 13.2.

Hence, we get $J(y_1, y_2, y_3, \ldots y_n)$

$$= \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3} \cdots \frac{\partial y_n}{\partial x_n}$$

$$= (-1)(-x_1)(-x_1 x_2) \ldots (-x_1 x_2 x_3 \ldots x_{n-1})$$

$$= (-1)^n x_1^{n-1} \cdot x_2^{n-2} \cdot \ldots x_{n-1}.$$

Ex.6. Find the Jacobian of $y_1, y_2, y_3, \ldots y_n$, being given

$$y_1 = x_1(1 - x_2), \; y_2 = x_1 x_2 (1 - x_3), \ldots$$

$$y_{n-1} = x_1 x_2 x_3 \ldots x_{n-1}(1 - x_n), y_n = x_1 x_2 x_3 \ldots x_n.$$

Soln. Adding all the given equations, we get

$$y_1 + y_2 + y_3 + \ldots + y_n = x_1.$$

Thus we have the following set of equations :

$$y_1 = x_1(1 - x_2), \; y_2 = x_1 x_2 (1 - x_3), \ldots$$

$$y_{n-1} = x_1 x_2 x_3 \ldots x_{n-1}(1 - x_n),$$

$$y_n = x_1 - y_1 - y_2 - \ldots - y_{n-1}.$$

$$\therefore \frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)}$$

$$= \begin{vmatrix} 1-x_2 & -x_1 & 0 & \ldots & 0 \\ x_2(1-x_3) & x_1(1-x_3) & -x_1x_2 & \ldots & 0 \\ x_2x_3(1-x_4) & x_1x_3(1-x_4) & x_1x_2(1-x_4) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \end{vmatrix}$$

$$= (-1)^{n-1}(-x_1)(-x_1x_2)(-x_1x_2x_3) \ldots (-x_1x_2 \ldots x_{n-1});$$

expanding in terms of the elements of the last row.

$$= (-1)^{n-1} \cdot (-1)^{n-1} \{x_1^{n-1} x_2^{n-2} \ldots x_{n-1}\}$$

$$= x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

Ex.7. If $u = \dfrac{x^2+y^2+z^2}{x}$, $v = \dfrac{x^2+y^2+z^2}{y}$, $w = \dfrac{x^2+y^2+z^2}{z}$,

find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

Soln. We have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1-\frac{y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix};$$

Applying $C_1 \to C_1 + \dfrac{y}{x} C_2 + \dfrac{z}{x} C_3$, we get

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{x^2+y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{x^2+y^2+z^2}{xy} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{x^2+y^2+z^2}{xz} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)}{x^2 \cdot xy \cdot xz} \begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy - \frac{x}{y}(x^2+z^2) & 2xz \\ 1 & 2xy & xz - \frac{x}{z}(x^2+y^2) \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)}{x^4yz} \begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\frac{x(x^2+y^2+z^2)}{y} & 0 \\ 0 & 0 & -\frac{x}{z}(x^2+y^2+z^2) \end{vmatrix}$$

applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \frac{(x^2+y^2+z^2)}{x^4yz} \times \left\{ \frac{(x(x^2+y^2+z^2)}{y} \cdot \frac{x}{z}(x^2+y^2+z^2) \right\}$$

$$= \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}.$$

Cor. : $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{x^2y^2z^2}{(x^2+y^2+z^2)^3}$; $\because \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \times \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 1.$

Ex.8. If $u_1 = \dfrac{x_1}{x_n}, u_2 = \dfrac{x_2}{x_n}, u_3 = \dfrac{x_3}{x_n}, \ldots u_{n-1} = \dfrac{x_{n-1}}{x_n}$,

and $x_1^2 + x_2^2 + x_3^2 + \ldots + x_{n-1}^2 + x_n^2 = 1$, **find the value of**

$$\frac{\partial(u_1, u_2, \ldots u_{n-1})}{\partial(x_1, x_2, \ldots x_{n-1})}.$$

Soln. Since x_n is a function of $x_1, x_2, \ldots x_{n-1}$ given by

$x_n^2 = 1 - x_1^2 - x_2^2 - \ldots - x_{n-1}^2$, we have

$$2x_n \frac{\partial x_n}{\partial x_1} = -2x_1 \Rightarrow \frac{\partial x_n}{\partial x_1} = -\frac{x_1}{x_n} \text{ etc.}$$

Therefore $\dfrac{\partial u_1}{\partial x_1} = \dfrac{x_n \cdot 1 - x_1 \dfrac{\partial x_n}{\partial x_1}}{x_n^2}$;

from the quotient rule of differentiation

$$= \frac{x_n + x_1 \cdot \dfrac{x_1}{x_n}}{x_n^2} = \frac{x_n^2 + x_1^2}{x_n^3};$$

and $\dfrac{\partial u_1}{\partial x_2} = x_1\left(-\dfrac{1}{x_n^{\,2}}\right)\cdot\dfrac{\partial x_n}{\partial x_2} = -\dfrac{x_1}{x_n^{\,2}}\cdot\left(-\dfrac{x_2}{x_n}\right) = \dfrac{x_1x_2}{x_n^{\,3}}$ etc.

Hence we have $\dfrac{\partial(u_1, u_2, \ldots u_{n-1})}{\partial(x_1, x_2, \ldots x_{n-1})}$

$$= \begin{vmatrix} \dfrac{x_n^{\,2}+x_1^{\,2}}{x_n^{\,3}} & \dfrac{x_1x_2}{x_n^{\,3}} & \dfrac{x_1x_3}{x_n^{\,3}} & \cdots & \dfrac{x_1x_{n-1}}{x_n^{\,3}} \\ \dfrac{x_2x_1}{x_n^{\,3}} & \dfrac{x_n^{\,2}+x_2^{\,2}}{x_n^{\,3}} & \dfrac{x_2x_3}{x_n^{\,3}} & \cdots & \dfrac{x_2x_{n-1}}{x_n^{\,3}} \\ \dfrac{x_3x_1}{x_n^{\,3}} & \dfrac{x_3x_2}{x_n^{\,3}} & \dfrac{x_n^{\,2}+x_3^{\,2}}{x_n^{\,3}} & \cdots & \dfrac{x_3x_{n-1}}{x_n^{\,3}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{x_{n-1}x_1}{x_n^{\,3}} & \dfrac{x_{n-1}x_2}{x_n^{\,3}} & \dfrac{x_{n-1}x_3}{x_n^{\,3}} & \cdots & \dfrac{x_n^{\,2}+x_{n-1}^{\,2}}{x_n^{\,3}} \end{vmatrix}$$

$$= \frac{1}{(x_n^{\,3})^{n-1}} \times$$

$$\begin{vmatrix} x_n^{\,2}+x_1^{\,2} & x_1x_2 & x_1x_3 & \cdots & x_1x_{n-1} \\ x_2x_1 & x_n^{\,2}+x_2^{\,2} & x_2x_3 & \cdots & x_2x_{n-1} \\ x_3x_1 & x_3x_2 & x_n^{\,2}+x_3^{\,2} & \cdots & x_3x_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1}x_1 & x_{n-1}x_2 & x_{n-1}x_3 & \cdots & x_n^{\,2}+x_{n-1}^{\,2} \end{vmatrix} \quad \ldots(1)$$

Multiplying C_1 by x_1, C_2 by x_2, C_3 by x_3, ... and adding to C_1 and using the relation $x_1^{\,2} + x_2^{\,2} + \ldots + x_n^{\,2} = 1$, we get

$$(1) = \frac{1}{x_n^{\,3n-3}\cdot x_1} \times$$

$$\begin{vmatrix} x_1 & x_1x_2 & x_1x_3 & \cdots & x_1x_{n-1} \\ x_2 & x_n^{\,2}+x_2^{\,2} & x_2x_3 & \cdots & x_2x_{n-1} \\ x_3 & x_3x_2 & x_n^{\,2}+x_3^{\,2} & \cdots & x_3x_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_{n-1}x_2 & x_{n-1}x_3 & \cdots & x_n^{\,2}+x_{n-1}^{\,2} \end{vmatrix}$$

$$= \frac{1}{x_n^{3n-3} \cdot x_1} \cdot x_1 \times \begin{vmatrix} 1 & x_2 & x_3 & \dots & x_{n-1} \\ x_2 & x_n^2 + x_2^2 & x_2x_3 & \dots & x_2x_{n-1} \\ x_3 & x_3x_2 & x_n^2 + x_3^2 & \dots & x_3x_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n-1} & x_{n-1}x_2 & x_{n-1}x_3 & \dots & x_n^2 + x_{n-1}^2 \end{vmatrix}$$

Now applying $R_2 \to R_2 - x_2R_1$, $R_3 \to R_3 - x_3R_1$, ...

$R_{n-1} \to R_{n-1} - x_{n-1}R_1$, we get

$$(1) = \frac{1}{x_n^{3n-3}} \begin{vmatrix} 1 & x_2 & x_3 & \dots & x_{n-1} \\ 0 & x_n^2 & 0 & \dots & 0 \\ 0 & 0 & x_n^2 & \dots & 0 \\ 0 & 0 & 0 & \dots & x_n^2 \end{vmatrix}$$

$$= \frac{1}{x_n^{3n-3}} \cdot (x_n^2)^{n-2} = \frac{x_n^{2n-4}}{x_n^{3n-3}} = \frac{1}{x_n^{n+1}}.$$

Hence the required Jacobian is $\frac{1}{x_n^{n+1}}$.

13.3 Properties of Jacobians

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable. A few of the important relations are given here and the proofs all depend upon the algebra of determinants. For simplicity, the properties are stated in terms of two or three variables only. But they are evidently true in general, however many variables may be involved.

Theorem I. If u_1, u_2, u_3 are functions of y_1, y_2, y_3 and y_1, y_2, y_3 are functions of x_1, x_2, x_3, then $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)} \times \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.

Proof : We have

$$\frac{\partial u}{\partial x_1} = \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial u_1}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_1} = \sum \frac{\partial u_1}{\partial y_r} \cdot \frac{\partial y_r}{\partial x_1}; r = 1, 2, 3$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_2} + \frac{\partial u_1}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_2} = \sum \frac{\partial u_1}{\partial y_r} \cdot \frac{\partial y_r}{\partial x_2}$$

$$\frac{\partial u_1}{\partial x_3}=\frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_3}+\frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_3}+\frac{\partial u_1}{\partial y_3}\cdot\frac{\partial y_3}{\partial x_3}=\sum\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_3}.$$

Similarly for $\frac{\partial u_2}{\partial x_1}, \ldots \frac{\partial u_3}{\partial x_3}, \ldots$ ...(A)

Now $\dfrac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)}\times\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$

$$=\begin{vmatrix}\frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3}\\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3}\\ \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3}\end{vmatrix}\times\begin{vmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3}\\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3}\end{vmatrix}$$

$$=\begin{vmatrix}\sum\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_2} & \sum\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_3}\\ \sum\frac{\partial u_2}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_2}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_2} & \sum\frac{\partial u_2}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_3}\\ \sum\frac{\partial u_3}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_3}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_2} & \sum\frac{\partial u_3}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_3}\end{vmatrix};$$

by the product of two determinants

$$=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{vmatrix};$$ using the relations (A)

$$=\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}.$$

Note The theorem can be extended upto n number of variables.

Theorem 2. To prove that $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}\times\dfrac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}=1.$

Proof : Let $y_1=f_1(x_1, x_2, x_3)$, $y_2=f_2(x_1, x_2, x_3)$ and $y_3=f_3(x_1, x_2, x_3)$.

...(A)

We assume the existence of inverse functions x_1, x_2, x_3 i.e. we assume that the equations which define y_1, y_2, y_3 as functions of x_1, x_2, x_3, determine x_1, x_2, x_3 as functions of y_1, y_2, y_3.

Let these relations may be put in the form

$x_1 = F_1(y_1, y_2, y_3), x_2 = F_2(y_1, y_2, y_3), x_3 = F_3(y_1, y_2, y_3)$.

Now, $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \times \dfrac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

$$= \begin{vmatrix} \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_3} \\ \sum \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \sum \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \sum \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_3} \\ \sum \frac{\partial y_3}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \sum \frac{\partial y_3}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \sum \frac{\partial y_3}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_3} \end{vmatrix};$$

$r = 1, 2, 3, \ldots$ ...(B)

But from (A),

$$1 = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_1} + \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_1} + \frac{\partial y_1}{\partial x_3} \cdot \frac{\partial x_3}{\partial y_1} = \sum \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial x_r}{\partial y_1}$$

$$0 = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_2} + \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_2} + \frac{\partial y_1}{\partial x_3} \cdot \frac{\partial x_3}{\partial y_2} = \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2}$$

Similarly, $1 = \sum \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} = \sum \frac{\partial y_3}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_3}$;

$$0 = \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} = \sum \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_3} \text{ etc.} \quad \ldots(C)$$

Thus using relations (C), (B) becomes

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \times \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Remark : This result is also true for n variables.

Note This may be regarded as the generalisation of the result

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

13.4 Jacobian of Implicit functions

As before, we shall confine ourselves to the three number of variables, though the treatment is quite general.

Let u_1, u_2, u_3 instead of being given explicitely in terms of x_1, x_2, x_3 be connected with them by equations such as

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0.$$

In this case, u_1, u_2, u_3 may be regarded as implicit functions of x_1, x_2, x_3.

On differentiation, we have

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0$$

$$\frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_2} = 0 \text{ etc.}$$

.....

$$\frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial F_2}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0 \text{ etc.} \quad \text{...(A)}$$

Now $\dfrac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)} \times \dfrac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$.

$$= \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \sum \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \sum \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial F_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \\ \sum \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \sum \frac{\partial F_3}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_3} \end{vmatrix};$$

$r = 1, 2, 3$

$$= \begin{vmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & -\frac{\partial F_1}{\partial x_3} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_3} \\ -\frac{\partial F_3}{\partial x_1} & -\frac{\partial F_3}{\partial x_2} & -\frac{\partial F_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}.$$

Note **Generalisation :** Let the number of variables be n, so that

$$\left.\begin{aligned} &F_1(u_1, u_2, u_3, \ldots u_n, x_1, x_2, x_3, \ldots x_n) = 0 \\ &F_2(u_1, u_2, u_3, \ldots u_n, x_1, x_2, x_3, \ldots x_n) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &F_n(u_1, u_2, u_3, \ldots u_n, x_1, x_2, x_3, \ldots x_n) = 0 \end{aligned}\right\} \quad \ldots(A)$$

Then $\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(u_1, u_2, \ldots u_n)} \times \dfrac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(x_1, x_2, \ldots x_n)}.$

Accordingly, we have

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \cdot \frac{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(x_1, x_2, \ldots x_n)}}{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(u_1, u_2, \ldots u_n)}}.$$

This result may be regarded as a generalisation of the result

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

where x and y are connected by the equation $f(x, y) = 0$.

Remark : The equations (A) of Art 13.4 connecting the variables are always capable of being transformed into the following form :

$$\phi_1(x_1, x_2, \ldots x_n, u_1) = 0$$

$$\phi_2(x_2, x_3, \ldots x_n, u_1, u_2) = 0$$

$$\phi_3(x_3, x_4, \ldots x_n, u_1, u_2, u_3) = 0$$

.....

$$\phi_n(x_n, u_1, u_2, \ldots u_n) = 0.$$

Now, $$\frac{\partial(\phi_1, \phi_2, \ldots \phi_n)}{\partial(u_1, u_2, \ldots u_n)} = \begin{vmatrix} \frac{\partial\phi_1}{\partial u_1} & \frac{\partial\phi_1}{\partial u_2} & \cdots & \frac{\partial\phi_1}{\partial u_n} \\ \frac{\partial\phi_2}{\partial u_1} & \frac{\partial\phi_2}{\partial u_2} & \cdots & \frac{\partial\phi_2}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\phi_n}{\partial u_1} & \frac{\partial\phi_n}{\partial u_2} & \cdots & \frac{\partial\phi_n}{\partial u_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial\phi_1}{\partial u_1} & 0 & \cdots & 0 \\ \frac{\partial\phi_2}{\partial u_1} & \frac{\partial\phi_2}{\partial u_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\phi_n}{\partial u_1} & \frac{\partial\phi_n}{\partial u_2} & \cdots & \frac{\partial\phi_n}{\partial u_n} \end{vmatrix}$$

$$= \frac{\partial\phi_1}{\partial u_1} \cdot \frac{\partial\phi_2}{\partial u_2} \cdots \frac{\partial\phi_n}{\partial u_n}.$$

Again, $$\frac{\partial(\phi_1, \phi_2, \ldots \phi_n)}{\partial(x_1, x_2, \ldots x_n)} = \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ 0 & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix}.$$

$$= \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial \phi_2}{\partial x_2} \cdots \frac{\partial \phi_n}{\partial x_n}.$$

Hence $$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \cdot \frac{\dfrac{\partial(\phi_1, \phi_2, \ldots \phi_n)}{\partial(x_1, x_2, \ldots x_n)}}{\dfrac{\partial(\phi_1, \phi_2, \ldots \phi_n)}{\partial(u_1, u_2, \ldots u_n)}}$$

$$= (-1)^n \cdot \frac{\dfrac{\partial \phi_1}{\partial x_1} \cdot \dfrac{\partial \phi_2}{\partial x_2} \cdots \dfrac{\partial \phi_n}{\partial x_n}}{\dfrac{\partial \phi_1}{\partial u_1} \cdot \dfrac{\partial \phi_2}{\partial u_2} \cdots \dfrac{\partial \phi_n}{\partial u_n}}.$$

Worked Out Examples

Ex.1. If $x + y + z = u$, $y + z = uv$, $z = uvw$, find the value of the Jacobian x, y, z with regard to u, v, w.

Soln. Let $F_1 \equiv x + y + z - u = 0$

$F_2 \equiv y + z - uv = 0$

$F_3 \equiv z - uvw = 0.$

We have, $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = -\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \times \frac{\partial(x, y, z)}{\partial(u, v, w)} \quad \ldots(1)$$

Now, $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -wu & -uv \end{vmatrix}$$

$$= -1 \begin{vmatrix} -u & 0 \\ -wu & -uv \end{vmatrix} = -u^2v \qquad \ldots(2)$$

Again, $$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \ldots(3)$$

Hence using (2) and (3), (1) becomes

$$-u^2v = -(1) \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

Thus $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v.$$

Ex.2. If $u = x(1 - r^2)^{-1/2}$, $v = y(1 - r^2)^{-1/2}$ and $w = z(1 - r^2)^{-1/2}$ where $r^2 = x^2 + y^2 + z^2$, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = (1 - r^2)^{-5/2}$.

Soln. We have, $x^2 = u^2(1 - r^2) = u^2(1 - x^2 - y^2 - z^2)$ etc.

Let $F_1 \equiv x^2 - u^2(1 - x^2 - y^2 - z^2) = 0$

$F_2 \equiv y^2 - v^2(1 - x^2 - y^2 - z^2) = 0$

$F_3 \equiv z^2 - w^2(1 - x^2 - y^2 - z^2) = 0.$

We have, $$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = (-1)^3 \times \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \times \frac{\partial(u, v, w)}{\partial(x, y, z)} \qquad \ldots(1)$$

Now, $$\frac{\partial(F_1, F_2, F_3)}{}$$

$$= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x + 2xu^2 & 2yu^2 & 2zu^2 \\ 2xv^2 & 2y + 2uv^2 & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z + 2zw^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2x(1+u^2) & 2yu^2 & 2zu^2 \\ 2xv^2 & 2y(1+v^2) & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z(1+w^2) \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1+u^2 & u^2 & u^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix};$$

applying $R_1 \to R_1 + R_2 + R_3$

$$= 8xyz(1+u^2+v^2+w^2) \begin{vmatrix} 1 & 1 & 1 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$$= 8xyz(1+u^2+v^2+w^2) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & v^2 \\ 0 & -1 & 1+w^2 \end{vmatrix};$$

applying $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$= 8xyz(1+u^2+v^2+w^2) \cdot 1$$

$$= 8xyz(1+u^2+v^2+w^2) \quad \text{...(2)}$$

Again, $\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} & \dfrac{\partial F_1}{\partial w} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} & \dfrac{\partial F_2}{\partial w} \\ \dfrac{\partial F_3}{\partial u} & \dfrac{\partial F_3}{\partial v} & \dfrac{\partial F_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} -2u(1-x^2-y^2-z^2) & 0 & 0 \\ 0 & -2v(1-x^2-y^2-z^2) & 0 \\ 0 & 0 & -2w(1-x^2-y^2-z^2) \end{vmatrix}$$

$$= \begin{vmatrix} -2u(1-r^2) & 0 & 0 \\ 0 & -2v(1-r^2) & 0 \\ 0 & 0 & -2w(1-r^2) \end{vmatrix}$$

$$= -8(1-r^2)^3 \begin{vmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{vmatrix}$$

$$= -8\,(1-r^2)^3(uvw) \qquad \ldots(3)$$

Hence using (2) and (3), (1) becomes

$$8xyz(1+u^2+v^2+w^2)$$

$$= -(-8)(1-r^2)^3(uvw) \times \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{xyz\,(1+u^2+v^2+w^2)}{uvw(1-r^2)^3} \qquad \ldots(4)$$

But $1+u^2+v^2+w^2 = 1+\dfrac{x^2+y^2+z^2}{1-r^2} = 1+\dfrac{r^2}{1-r^2} = \dfrac{1}{1-r^2}$

and $uvw = xyz = (1-r^2)^{-3/2}$.

Hence from (4),

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{xyz \cdot \dfrac{1}{1-r^2}}{xyz(1-r^2)^{-3/2}(1-r^2)^3}$$

$$= \frac{1}{(1-r^2)^{-3/2}(1-r^2)^4}$$

$$= \frac{1}{(1-r^2)^{4-3/2}}$$

$$= \frac{1}{(1-r^2)^{5/2}} = (1-r^2)^{-5/2}.$$

Ex.3. The roots of the equation in λ, viz $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ are u, v, w.

Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = -2\dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$.

Soln. The equation in λ can be written as

$$3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0.$$

Since u, v, w are the roots of the above equation, we have

$$u+v+w = x+y+z,$$

$$uv+vw+wu = x^2+y^2+z^2$$

and $$uvw = \frac{1}{3}(x^3+y^3+z^3).$$

Now, let $F_1 \equiv u+v+w-x-y-z=0$

$$F_2 \equiv uv+vw+wu-x^2-y^2-z^2=0$$

$$F_3 \equiv uvw - \frac{1}{3}(x^3+y^3+z^3) = 0.$$

We have, $$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = (-1)^3 \cdot \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} \quad ...(1)$$

Now, $$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)^3 \cdot 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -2(y-z)(z-x)(x-y) \qquad \ldots(2)$$

Again as above, $\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix};$$

applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= (u-v)v(u-w) - (u-w)\,w(u-v)$$

$$= (u-v)(u-w)(v-w)$$

$$= -(u-v)(v-w)(w-u) \qquad \ldots(3)$$

Hence using (2) and (3), (1) becomes

$$-2(y-z)(z-x)(x-y) = +(u-v)(v-w)(w-u) \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$\Rightarrow \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{-2(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$

13.5 Necessary and sufficient condition for a Jacobian to vanish

Theorem : Let $u_1, u_2, u_3, \ldots u_n$ be functions of n independent variables $x_1, x_2, x_3, \ldots x_n$. In order that there may exist between these functions a relation $F(u_1, u_2, u_3, \ldots u_n) = 0$, it is necessary and sufficient that the Jacobian $\dfrac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$ should vanish identically.

Proof : The condition is necessary.

Let there exists between $u_1, u_2, u_3, \ldots u_n$ a relation

$$F(u_1, u_2, \ldots u_n) = 0. \qquad \ldots(1)$$

We shall show that their Jacobian is zero.

Differentiating (1), we get

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1}+\ldots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_1}=0$$

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}+\ldots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_2}=0$$

.....

$$\frac{\partial F}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_n}+\frac{\partial F}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_n}+\ldots+\frac{\partial F}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_n}=0.$$

Eliminating $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \ldots \frac{\partial F}{\partial u_n}$ from the above set of equations, we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \ldots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_n}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \ldots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_2}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0;$$

by interchanging the rows and columns

i.e. $$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = 0.$$

The condition is sufficient

We shall prove that if the Jacobian $\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$ is zero, then there exists a relations between $u_1, u_2, \ldots u_n$.

The equations connecting the functions $u_1, u_2, \ldots u_n$ and the variables $x_1, x_2, \ldots x_n$ are always capable of being written in the following forms :

$$\phi_1(x_1, x_2, \ldots x_n, u_1) = 0$$
$$\phi_2(x_2, x_3, \ldots x_n, u_1, u_2) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi_r(x_r, x_{r+1}, \ldots x_n, u_1, u_2, \ldots u_r) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi_n(x_n, u_1, u_2, u_3, \ldots u_n) = 0.$$

Then by the remark of Art 13.4, we have

$$J = \frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\dfrac{\partial\phi_1}{\partial x_1} \cdot \dfrac{\partial\phi_2}{\partial x_2} \cdots \dfrac{\partial\phi_r}{\partial x_r} \cdots \dfrac{\partial\phi_n}{\partial x_n}}{\dfrac{\partial\phi_1}{\partial u_1} \cdot \dfrac{\partial\phi_2}{\partial u_2} \cdots \dfrac{\partial\phi_n}{\partial u_n}}.$$

Now, if $J = 0$, we have

$$\frac{\partial\phi_1}{\partial x_1} \cdot \frac{\partial\phi_2}{\partial x_2} \cdots \frac{\partial\phi_r}{\partial x_r} \cdots \frac{\partial\phi_n}{\partial x_n} = 0.$$

i.e. $\dfrac{\partial\phi_r}{\partial x_r} = 0$ for some value of r between 1 and n.

Hence for that particular value of r, the function ϕ_r must not contain x_r.

Accordingly the corresponding equation is of the form

$$\phi_r(x_{r+1}, \ldots x_n, u_1, u_2, \ldots u_n) = 0 \qquad \ldots(1)$$

Consequently, the variables $x_{r+1}, x_{r+2}, \ldots x_n$ can be eliminated between the equation (1) and the remaining equations $\phi_{r+1} = 0$, $\phi_{r+2} = 0, \ldots \phi_n = 0$ so as to give a final equation between $u_1, u_2, \ldots u_n$ alone.

Hence the theorem is proved.

Ex.1. If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$,

show that the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes identically.

Soln. We have, $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix};$$

applying $R_1 \to R_1 + R_3$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$= 2(x+y+z) \times 0$

$= 0$; since R_1 and R_2 are identical.

Note In order to find the relation between u, v, w, we have

$$v^2 = (x+y+z)^2 = (x^2+y^2+z^2) + 2(xy+yz+zx) = u + 2w.$$

Thus $v^2 = u + 2w$ which is the relation between u, v and w.

Ex.2. Show that the functions

$$u = x+y+z,\ v = x^2+y^2+z^2,\ w = x^3+y^3+z^3-3xyz$$

are not independent. Find the relation between them.

Soln. We have, $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= 2 \times 3 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= 6 \begin{vmatrix} 0 & 0 & 1 \\ x - y & y - z & z \\ x^2 - yz - y^2 + zx & y^2 - zx - z^2 + xy & z^2 - xy \end{vmatrix};$$

applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= 6 \begin{vmatrix} x - y & y - z \\ (x^2 - y^2) + (zx - yz) & (y^2 - z^2) + (xy - zx) \end{vmatrix}$$

$$= 6 \begin{vmatrix} x - y & y - z \\ (x - y)(x + y) + z(x - y) & (y - z)(y + z) + x(y - z) \end{vmatrix}$$

$$= 6 \begin{vmatrix} x - y & y - z \\ (x - y)(x + y + z) & (y - z)(y + z + x) \end{vmatrix}$$

$$= 6(x - y)(y - z) \begin{vmatrix} 1 & 1 \\ x + y + z & x + y + z \end{vmatrix}$$

$= 6(x - y)(y - z) \times 0$; since two columns are identical

$= 0$.

Hence u, v, w are not independent. To find a relation between u, v, w, we have

$$u^3 = (x + y + z)^3$$
$$= \Sigma x^3 + 3\Sigma xy^2 + 6xyz \qquad \ldots(1)$$

Now, $(x + y + z)(x^2 + y^2 + z^2) = \Sigma x^3 + \Sigma xy^2$

$\Rightarrow \quad u \cdot v = \Sigma x^3 + \Sigma xy^2$

$\Rightarrow \quad 3uv = 3\Sigma x^3 + 3\Sigma xy^2$.

Hence from (1), $u^3 = \Sigma x^3 + (3uv - 3\Sigma x^3) + 6xyz$

$$= 3uv - 2\Sigma x^3 + 6xyz$$
$$= 3uv - 2(\Sigma x^3 - 3xyz) = 3uv - 2w.$$

Thus $u^3 = 3uv - 2w$ is the required relation.

Ex.3. If $f(0)=0$ and $f'(x)=\dfrac{1}{1+x^2}$, prove without using the method of integration, that $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$.

Soln. Let $u=f(x)+f(y)$ and $v=\dfrac{x+y}{1-xy}$.

$$\text{Now } \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} f'(x) & f'(y) \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$=\frac{1}{(1-xy)^2}\begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ 1+y^2 & 1+x^2 \end{vmatrix}$$

$$=\frac{1}{(1-xy)^2}\left[\frac{1}{1+x^2}\cdot(1+x^2)-\frac{1}{1+y^2}\cdot(1+y^2)\right]$$

$$=\frac{1}{(1-xy)^2}[1-1]=0.$$

Hence u and v are connected by a functional relation.

Let $u=\phi(v)$.

i.e. $f(x)+f(y)=\phi\left(\dfrac{x+y}{1-xy}\right)$ ...(1)

Putting $y=0$, we get $f(x)+f(0)=\phi(x)$

$\Rightarrow \quad f(x)=\phi(x)$; since $f(0)=0$.

Thus from (1), $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$.

Exercise 13

1. If $x = u(1 + v)$ and $y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.

2. If $x = u(1 - v)$, $y = uv$, evaluate $J = \dfrac{\partial(x, y)}{\partial(u, v)}$ and $J' = \dfrac{\partial(u, v)}{\partial(x, y)}$ and hence verify that $JJ' = 1$.

3. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\dfrac{\partial(u, v)}{\partial(r, \theta)}$.

   [**Hint :** $\dfrac{\partial(u, v)}{\partial(r, \theta)} = \dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(r, \theta)}$.]

4. If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$, ... $y_n = \sin x_1 \sin x_2 \ldots \sin x_{n-1} \cos x_n$, show that the Jacobian of $y_1, y_2, \ldots y_n$ with respect to $x_1, x_2, \ldots x_n$ is $(-1)^n \sin^n x_1 \sin^{n-1} x_2, .. \sin x_n$.

5. If $x_1 + x_2 + \ldots + x_n = y_1$, $x_2 + x_3 + \ldots + x_n = y_1 y_2$, ... $x_n = y_1 y_2 \ldots y_n$, show that
$$\frac{\partial(x_1, x_2, \ldots x_n)}{\partial(y_1, y_2, \ldots y_n)} = y_1^{n-1} y_2^{n-2} \ldots y_{n-2}^2 y_{n-1}.$$

6. If $u_r = x_r(1 - x_1^2 - x_2^2 - \ldots - x_n^2)^{-1/2}$, $r = 1, 2, 3, \ldots n$ prove that $\dfrac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (1 - x_1^2 - x_2^2 - \ldots - x_n^2)^{-(n+2)/2}$.

7. If λ, μ, ν are the roots of the equation in k,
$$\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1,$$
prove that $\dfrac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)} = -\dfrac{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)}{(b - c)(c - a)(a - b)}$.

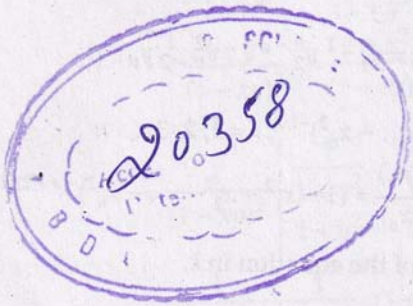
8. Show that the functions
$$u = x + y + z,\ v = xy + yz + zx,\ w = x^3 + y^3 + z^3 - 3xyz$$
are not independent. Find the relation between them.

9. If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w.

10. Show that the functions $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent. Find the relation between them.

11. Prove that the three functions u, v, w are connected by a functional relation if $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$.

ANSWERS

1. $1 + u + v$ 3. $4r^2$ 8. $u^3 = 3uv + w$
9. $u^2 - v^2 = 4w$ 10. $u^2 - v^2 = 8w$
11. $uv + vw + wu = -1$.

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